
CHAPTER 25: JOINT DENSITIES

Recall from Chapter 24, that the probability distribution, or **probability density function (pdf)**, of a continuous random variable X is a function $f_X(x)$, such that for all real values a, b with $a \leq b$,

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x) dx.$$

How to define the joint pdf for continuous r.v.'s?

Remarks:

- (1) Note that $f_{X,Y}(x, y) \neq \mathbb{P}(X = x, Y = y)$!!!
- (2) In order for $f_{X,Y}(x, y)$ to be a pdf, it needs to satisfy the properties
 - $f_{X,Y}(x, y) \geq 0$ for all x, y
 - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$

Double Integrals Mini Lesson

Example 25.1. *Solve the following integrals.*

(1) $\int_2^3 \int_0^1 xy dy dx$

(2) $\int_2^3 \int_0^1 (x + y) dy dx$

(3) $\int_2^3 \int_0^1 e^{x+y} dy dx$

Definition 25.2 (Joint cumulative distribution function).

The **joint cumulative distribution function (cdf)** of continuous random variables X and Y , is the function $F_{X,Y}(x, y)$, such that for all real values of x and y ,

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(s, t) dt ds$$

Remarks:

- The definition above for $F_{X,Y}(x, y)$ is a **function** of x and y .
- The joint cdf at the point (a, b) , is

$$F_{X,Y}(a, b) = \mathbb{P}(X \leq a, Y \leq b) = \int_{-\infty}^a \int_{-\infty}^b f_{X,Y}(s, t) dt ds$$

Definition 25.3 (Marginal pdf's).

Suppose X and Y are continuous r.v.'s, with joint pdf $f_{X,Y}(x, y)$. Then the **marginal probability density functions** are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$$

Example 25.4. Let $f_{X,Y}(x, y) = \frac{3}{2}y^2$, for $0 \leq x \leq 2$, $0 \leq y \leq 1$.

(1) Find $\mathbb{P}(0 \leq X \leq 1, 0 \leq Y \leq \frac{1}{2})$.

(2) Find $f_X(x)$ and $f_Y(y)$.

Example 25.5. Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

(1) Find $f_X(x)$ and $f_Y(y)$.

(2) Find $\mathbb{P}(Y < 3)$.

Example 25.6. Let X and Y have constant density on the square $0 \leq X \leq 4, 0 \leq Y \leq 4$.

(1) Find $\mathbb{P}(|X - Y| < 2)$.

Example 25.6 continued.

(2) *Let $M = \max(X, Y)$. Find the pdf for M , that is $f_M(m)$.*

(3) *Let $Z = \min(X, Y)$. Find the pdf for Z , that is $f_Z(z)$.*

Example 25.7. Let X and Y have joint density $f_{X,Y}(x,y) = \frac{8}{5}(x+y)$ in the region $0 < x < 1$, $\frac{1}{2} < y < 1$. Find the pdf of the r.v. Z , where $Z = XY$.

Example 25.7 solution continued.