Lesson 15: Inference for a single proportion or difference of two (independent) proportions

TB sections 8.1-8.2

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Learning Objectives

1. Remind ourselves of the Normal approximation of the binomial distribution and define the sampling distribution of a sample proportion
2. Run a hypothesis test for a single proportion and interpret the results.
3. Construct and interpret confidence intervals for a single proportion.

single prop

4. Understand how CLT applies to a difference in binomial random variables
5. Run a hypothesis test for a difference in proportions and interpret the results.
6. Construct and interpret confidence intervals for a difference in proportions.

diff in props

Where are we?

Data
- Collecting data
- Categorical vs. Numeric
- Summary statistics
- Data visualization

Probability
- Probability rules
- Independence, conditional
- Random variables and probability distributions
- Linear combinations
- Binomial, Normal, and Poisson

Sampling Variability, and Statistical Inference
- Sampling distributions
- Central Limit Theorem
- Confidence Intervals
- Hypothesis tests

Inference for continuous data/outcomes
- One sample t-test
- 3+ independent samples
- Simple linear regression / correlation
- 2 sample tests: paired and independent
- Power and sample size
- Non-parametric tests

Inference for categorical data/outcomes
- One proportion test
- Fisher's exact test
- Non-parametric tests
- Chi-squared test
- 2 proportion test
- Power and sample size

R
- Basics
- Reproducibility
- Quarto
- Packages
- Data visualization
- Data wrangling
- R Projects
- …

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Moving to categorical outcomes

- Previously, we have discussed methods of inference for numerical data
  - Our outcomes were numerical values
  - We were doing inference of **means**
  - We found confidence intervals for means
  - We ran hypothesis tests for means

- Above methods used can be extended to **categorical data**, such as binomial **proportions** or data in two-way tables

- **Categorical data arise frequently in medical research**
  - Disease outcomes and patient characteristics are often recorded in natural categories
  - **Examples:** types of treatment received, whether or not disease advanced to a later stage, or whether or not a patient responded initially to a treatment

From Lesson 5: Binomial random variable

- **One specific type of discrete random variable** is a binomial random variable

Binomial random variable

- X is a binomial random variable if it represents the number of successes in n independent replications (or trials) of an experiment where
  - Each replicate has two possible outcomes: either **success** or **failure**
  - The probability of success is p
  - The probability of failure is $q = 1 - p$

X is a binomial RV w/ n trials & probab of success p

- A binomial random variable takes on values $0, 1, 2, \ldots, n$.
- If a r.v. X is modeled by a Binomial distribution, then we write in shorthand $X \sim \text{Binom}(n, p)$
- Quick example: The number of heads in 3 tosses of a fair coin is a binomial random variable with parameters $n = 3$ and $p = 0.5$.

From Lesson 5: Binomial distribution

Distribution of a **Binomial** random variable

Let X be the total number of successes in n independent trials, each with probability p of a success. Then probability of observing exactly k successes in n independent trials is

$$\underset{0,1,2,\ldots,n}{P(X=x)} = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, 2, \ldots, n$$

- The parameters of a binomial distribution are $\underline{p \text{ and } n}$.
- If a r.v. X is modeled by a binomial distribution, then we write in shorthand $\underline{X \sim \text{Binom}(n, p)}$

Mean and variance of a Binomial r.v

If X is a binomial r.v. with probability of success p, then $E(X) = np$ and $\text{Var}(X) = np(1-p)$

From Lesson 6: Normal Approximation of the Binomial Distribution

- Also known as: **Sampling distribution of $\widehat{p}$**
- If $X \sim \text{Binomial}(n, p)$ and $np > 10$ and $nq = n(1-p) > 10$
  - Ensures sample size (n) is moderately large and the p is not too close to 0 or 1
  - Other resources use other criteria (like $npq > 5$ or $np > 5$)

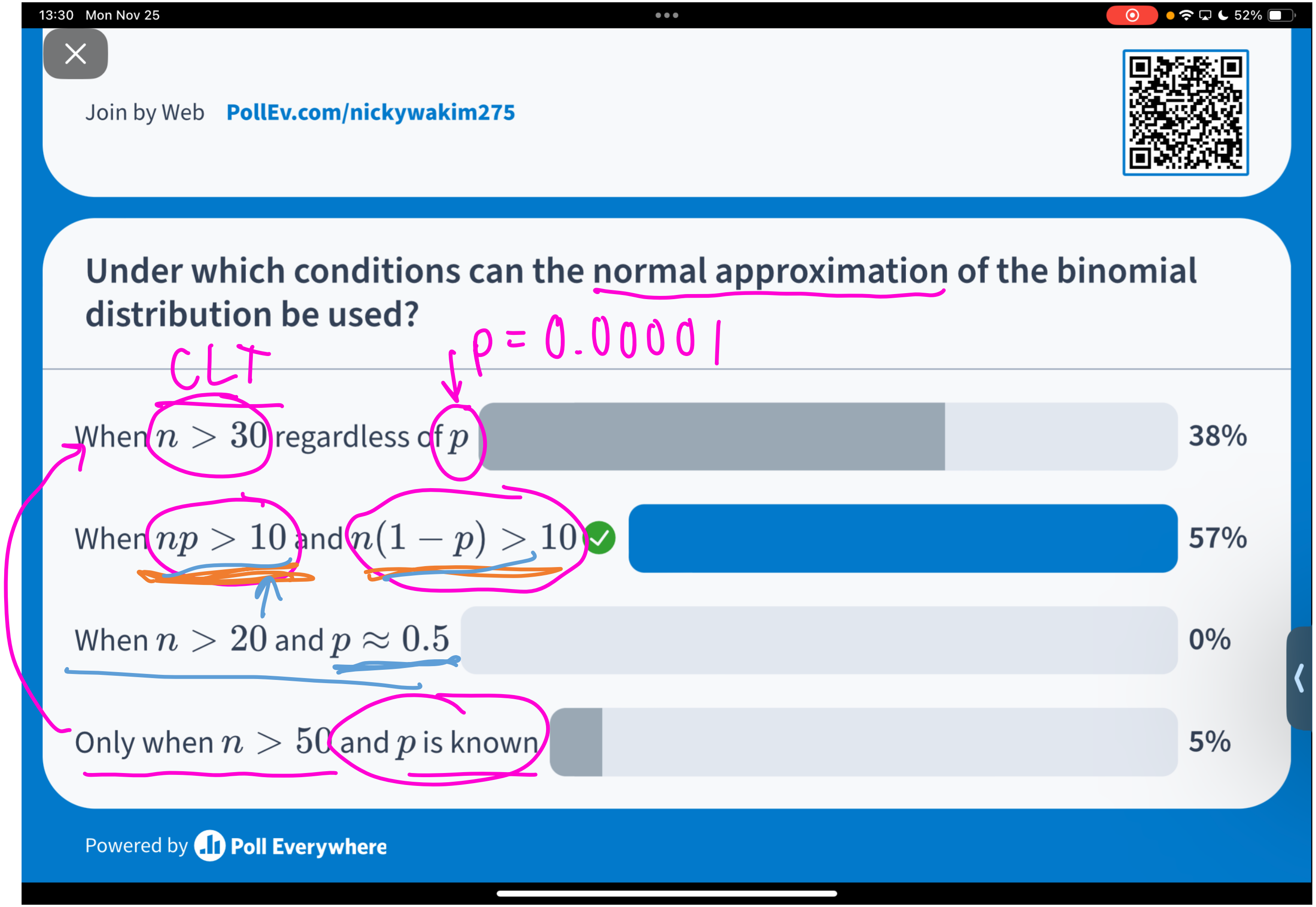
$\sqrt{\underbrace{np(1-p)}_{Var(X)}}$

- THEN approximately

$$X \sim \text{Normal}\left(\mu_X = \overset{E(X)}{\underline{np}}, \sigma_X = \sqrt{\underline{np(1-p)}}\right)$$

- **Continuity Correction:** Applied to account for the fact that the binomial distribution is discrete, while the normal distribution is continuous
  - Adjust the binomial value (# of successes) by ±0.5 before calculating the normal probability.
  - For $P(X \leq k)$ (Binomial), you would instead calculate $P(X \leq k + 0.5)$ (Normal approx)
  - For $P(X \geq k)$ (Binomial), you would instead calculate $P(X \leq k - 0.5)$ (Normal approx)

Poll Everywhere Question 1



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Under which conditions can the normal approximation of the binomial distribution be used?

p = 0.00001

CLT

When $n > 30$ regardless of p	38%
When $np > 10$ and $n(1-p) > 10$ ✓	57%
When $n > 20$ and $p \approx 0.5$	0%
Only when $n > 50$ and p is known	5%

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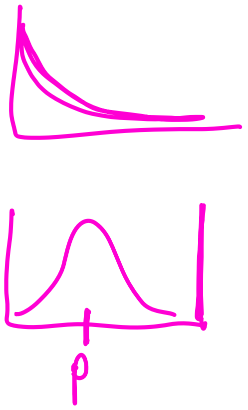


take a look @ Lesson 6

Sampling distribution of $\hat{p}$

- $\hat{p} = \frac{X}{n}$ where X is the number of "successes" and n is the sample size.
- $X \sim Bin(n, p)$, where p is the population proportion.
- For n "big enough", the normal distribution can be used to approximate a binomial distribution:

$$X \sim N\left(\mu = np, \sigma = \sqrt{np(1-p)}\right)$$

$$E\left(\frac{X}{n}\right) = \frac{1}{n}E(X)$$

- Since $\hat{p} = \frac{X}{n}$ is a linear transformation of X, we have for large n:

$$Var\left(\frac{X}{n}\right) = \frac{1}{n^2}Var(X)$$

$$\hat{p} \sim N\left(\mu_{\hat{p}} = p, \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}\right)$$

$$\frac{np(1-p)}{n^2}$$

- What is "big enough"? At least 10 successes and 10 failures are expected in the sample: $np \geq 10$ and $n(1-p) \geq 10$

For proportions: Population parameters vs. sample statistics

Population parameter ✓

- Proportion: p, π (“pi”)

Sample statistic (point estimate) ✓

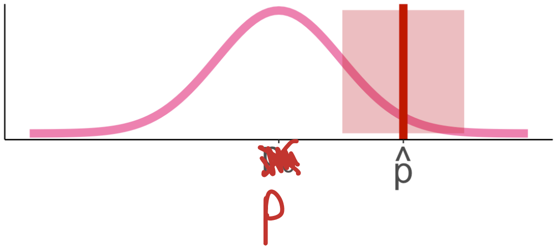
- Sample proportion: $\hat{p}$ (“p-hat”)

Approaches to answer a research question

- **Research question is a generic form for a single proportion:** Is there evidence to support that the population proportion is different than p_0?

null value

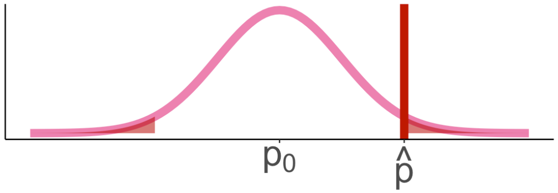
Calculate **CI for the proportion** p:



$$\hat{p} \pm z^* \cdot SE_{\hat{p}} = \hat{p} \pm z^* \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- with z^* = z-score that aligns with specific confidence interval

Run a **hypothesis test:**



Hypotheses

$$H_0: p = p_0$$
$$H_A: p \neq p_0$$
$$(or \;\; <, >)$$

Test statistic

$$z_{\hat{p}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \cdot (1-p_0)}{n}}}$$

R code: 1- and 2-sample proportions tests

```
prop.test(x,
          n,
          p = NULL,
          alternative = c("two.sided", "less", "greater"),
          conf.level = 0.95,
          correct = TRUE)
```

- `x`: Counts of successes (can have one x or a vector of multiple x's)
- `n`: Number of trails (can have one n or a vector of multiple n's)
- `p`: Null value that we think the population proportion is →

$$H_0: p = p_0$$

- `alternative`: If alternative hypothesis is $\neq$, $<$, or $>$
  - Default is "two.sided" ($\neq$)
- `conf.level` = Confidence level $(1-\alpha)$
  - Default is `0.05`
- `correct`: Continuity correction, whether we should use it or not
  - Default is `TRUE` (Nicky says keep it this way!)

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Example: immune response to advanced melanoma

- Looking for therapies that trigger an immune response to advanced melanoma
- In a study where 52 patients were treated concurrently with two new therapies, nivolumab and ipilimumab
    - 21 had an immune response.[1]
- **Outcome:** whether or not each person has an immune response

Questions that can be addressed with inference…

- What is the estimated population probability of immune response following concurrent therapy with nivolumab and ipilimumab? (calculate $\hat{p}$)
- What is the 95% confidence interval for the estimated population probability of immune response following concurrent therapy with nivolumab and ipilimumab? (95% CI of p)
- In previous studies, the proportion of patients responding to one of these agents was 30% or less. Do these results suggest that the probability of response to concurrent therapy is better than 0.30? (Hypothesis test of null of 0.3)

Reference: Steps in a Hypothesis Test

1. Check the **assumptions**
2. Set the **level of significance** α
3. Specify the **null** (H_0) and **alternative** (H_A) **hypotheses**
    1. In symbols
    2. In words
    3. Alternative: one- or two-sided?
4. Calculate the **test statistic**.
5. Calculate the **p-value** based on the observed test statistic and its sampling distribution
6. Write a **conclusion** to the hypothesis test
    1. Do we reject or fail to reject H_0?
    2. Write a conclusion in the context of the problem

Step 1: Check the assumptions (easier to do after Step 3)

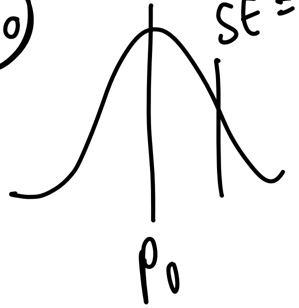
The sampling distribution of $\hat{p}$ is approximately normal when

1. The sample observations are independent, and ✓
2. At least 10 successes and 10 failures are expected in the sample: $np_0 \geq 10$ and $n(1 - p_0) \geq 10$.

- Since p is unknown, it is necessary to substitute p_0 (the null value) for p when using the standard error to conduct hypothesis tests
  - Because we are assuming the standard error of the null hypothesis!

$H_0: p = p_0$

$SE = \sqrt{\frac{p_0(1-p_0)}{n}}$



- For the example, we have $\boxed{p_0 = 0.30}$
  - We check: $np_0 = \underline{52} \cdot \underline{0.3} = \underline{15.6} > 10$
  - We check: $\underline{n(1 - p_0)} = 52(1 - 0.3) = \underline{36.4 > 10}$

Step 2: Set the level of significance

- **Before doing a hypothesis test**, we set a cut-off for how small the p-value should be in order to reject H_0.
- Typically choose $\alpha = 0.05$

- See Lesson 11: Hypothesis Testing 1: Single-sample mean

Step 3: Null & Alternative Hypotheses (1/2)

Notation for hypotheses (for paired data)

$$\begin{aligned} H_0 &: p = p_0 \\ \text{vs. } H_A &: p \neq, <, \text{or}, > p_0 \end{aligned}$$

Hypotheses test for example

$$\begin{aligned} H_0 &: p = 0.30 \\ \text{vs. } H_A &: p \neq 0.30 \end{aligned}$$

We call p_0 the *null value* (hypothesized population mean difference from H_0)

$H_A : p \neq p_0$

- not choosing a priori whether we believe the population proportion is greater or less than the null value p_0

$H_A : p < p_0$

- believe the population proportion is **less** than the null value p_0

$H_A : p > p_0$

- believe the population population proportion is **greater** than the null value p_0

- $H_A : p \neq p_0$ is the most common option, since it's the most conservative

Step 3: Null & Alternative Hypotheses (2/2)

Null and alternative hypotheses in **words** and in **symbols**.

One sample test

- H_0: For individuals who have advanced melanoma and received a treatment of nivolumab and ipilimumab, the population proportion of immune response is 0.30
- H_A: For individuals who have advanced melanoma and received a treatment of nivolumab and ipilimumab, the population proportion of immune response is NOT 0.30

$$\begin{aligned} H_0 &: p = 0.30 \\ H_A &: p \neq 0.30 \end{aligned}$$

Step 4: Test statistic

under null:

Sampling distribution of $\hat{p}$ if we assume $H_0 : p = p_0$ is true:

$$\hat{p} \sim N\left(\mu_{\hat{p}} = p, \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}\right) \sim N\left(\mu_{\hat{p}} = \underline{p_0}, \sigma_{\hat{p}} = \sqrt{\frac{\underline{p_0} \cdot (1 - \underline{p_0})}{n}}\right)$$

Test statistic for a one sample proportion test:

$$\text{test stat} = \frac{\text{point estimate} - \text{null value}}{SE}$$

$$z_{\hat{p}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \cdot (1 - p_0)}{n}}}$$

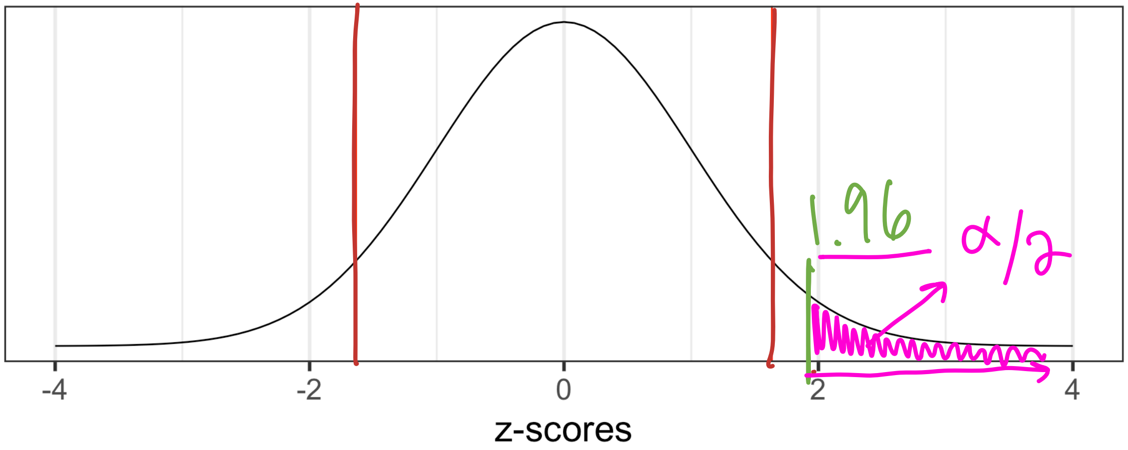
Step 4: Test statistic

From our example: Recall that $\hat{p} = \frac{21}{52} = 0.4038, n = 52$, and $p_0 = 0.30$

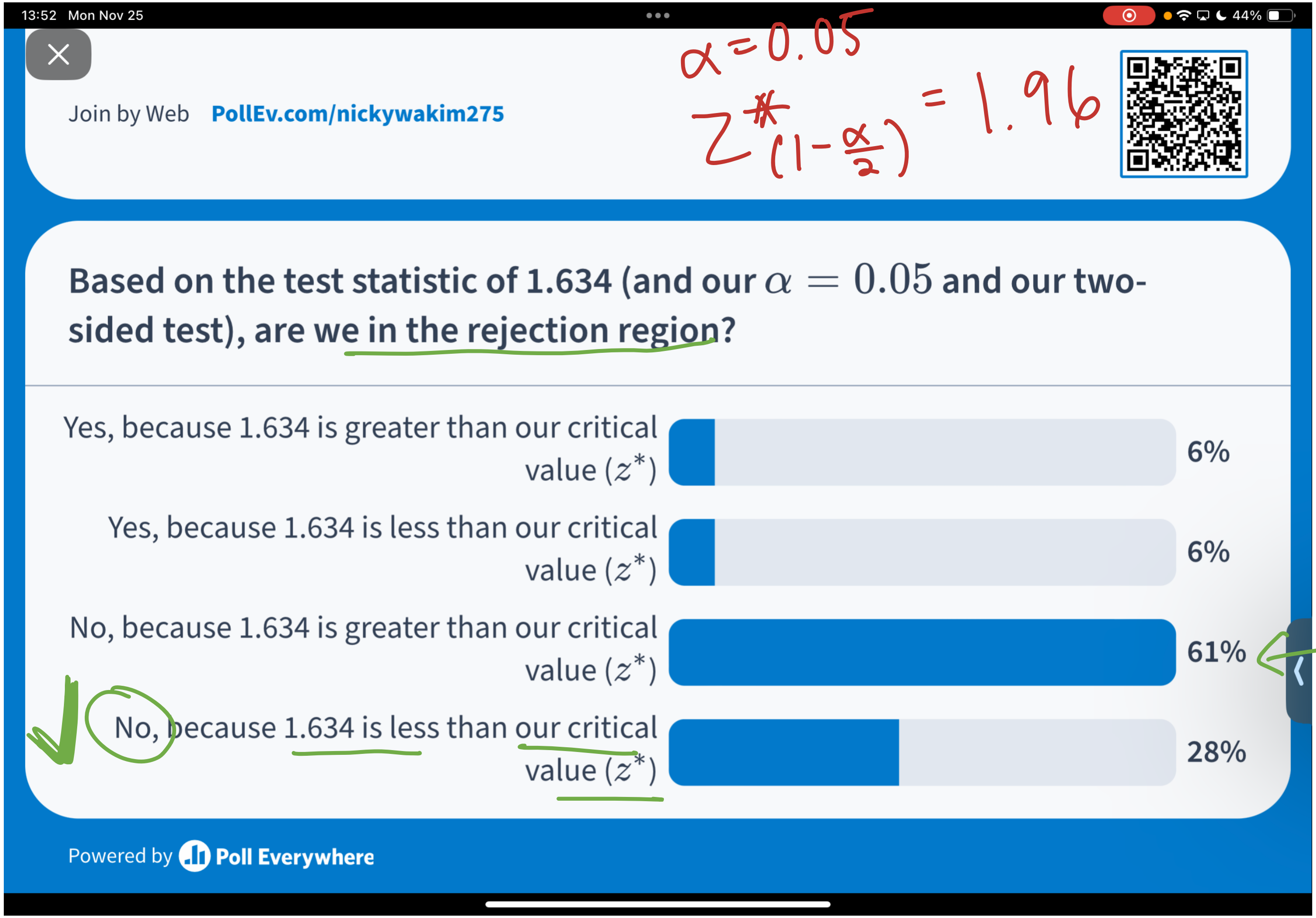
The test statistic is:

$$z_{\hat{p}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \cdot (1 - p_0)}{n}}} = \frac{21/52 - 0.30}{\sqrt{\frac{0.30 \cdot (1 - 0.30)}{52}}} = 1.6341143$$

- Let's see the z-score on a Z-distribution (Standard Normal curve)



Poll Everywhere Question 2



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$\alpha = 0.05$

$z^*_{(1-\frac{\alpha}{2})} = 1.96$

Based on the test statistic of 1.634 (and our $\alpha = 0.05$ and our two-sided test), are we in the rejection region?

Yes, because 1.634 is greater than our critical value (z^*)	6%
Yes, because 1.634 is less than our critical value (z^*)	6%
No, because 1.634 is greater than our critical value (z^*)	61%
[✓ No], because 1.634 is less than our critical value (z^*)	28%

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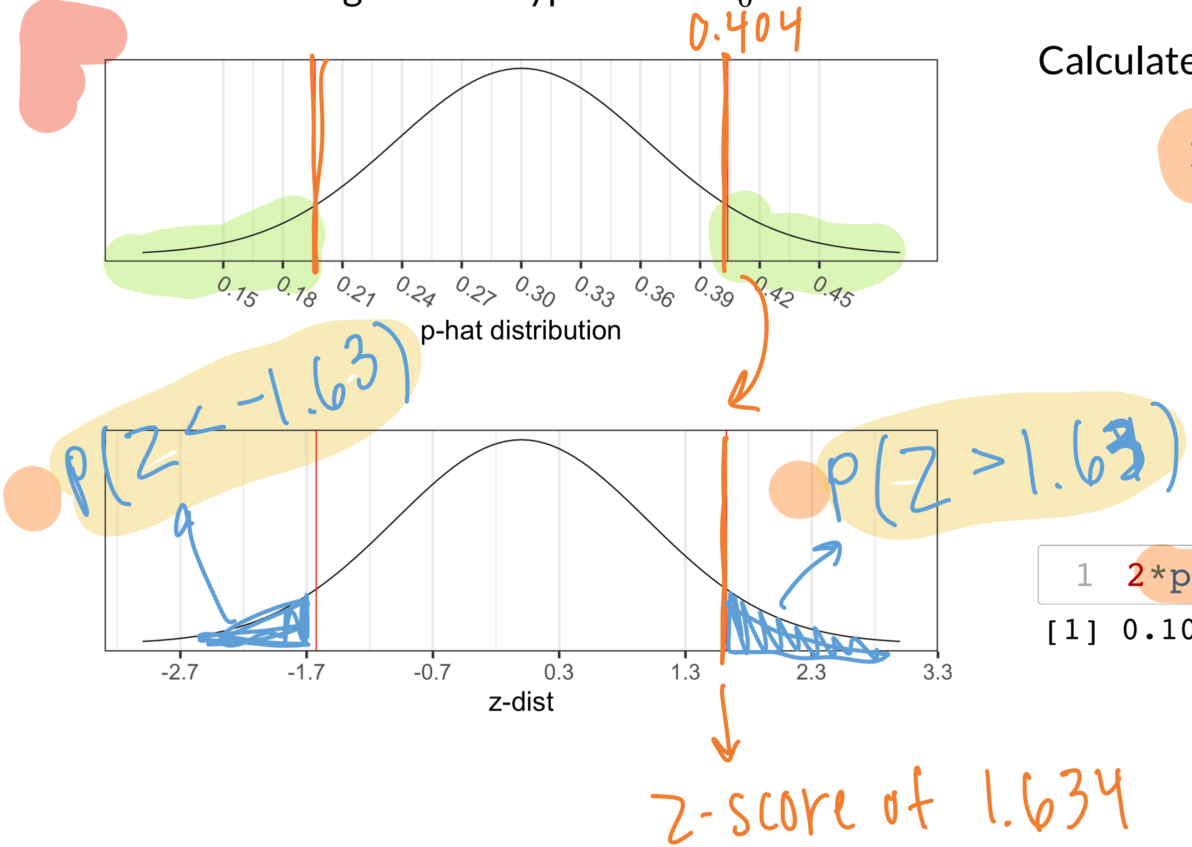
Step 5: p-value

The **p-value** is the **probability** of obtaining a test statistic *just as extreme or more extreme* than the observed test statistic assuming the null hypothesis H_0 is true.



Calculate the *p*-value:

$$2 \cdot P(\hat{p} > 0.404)$$

$$= 2 \cdot P\left(Z_{\hat{p}} > \frac{0.404 - 0.30}{\sqrt{\frac{0.30 \cdot (1 - 0.30)}{52}}}\right)$$

$$= 2 \cdot P(Z_{\hat{p}} > 1.634)$$

$$= 0.1022348$$

```r
2*pnorm(1.634, lower.tail = F)
```

```
[1] 0.1022589
```

Step 4-5: test statistic and p-value together using `prop.test()`

```
prop.test(x = 21, n = 52, p = 0.30, correct = T)
```

~~correct = T~~

succ

↳ # trials

p_0

```
    1-sample proportions test with continuity correction

data:  21 out of 52, null probability 0.3
X-squared = 2.1987, df = 1, p-value = 0.1381
alternative hypothesis: true p is not equal to 0.3
95 percent confidence interval:
 0.2731269 0.5487141
sample estimates:
        p
0.4038462
```

▶ Tidying the output of `prop.test()`

estimate	statistic	p.value	parameter	conf.low	conf.high	method	alternative
0.4038462	2.198718	0.1381256	1	0.2731269	0.5487141	1-sample proportions test with continuity correction	two.sided

- Note: We expect some differences between the test statistic and p-value calculated by hand vs. by R. R uses a slightly different method to calculate.

Step 6: Conclusion to hypothesis test

$$\begin{aligned} H_0 &: p = 0.30 \\ H_A &: p \neq 0.30 \end{aligned}$$

- Recall the p-value = 0.1022348
- Use α = 0.05.
- Do we reject or fail to reject H_0?

Conclusion statement:

- Stats class conclusion
  - There is insufficient evidence that the (population) proportion of individuals who had an immune response is different than 0.30 (p-value = 0.102).
- More realistic manuscript conclusion:
  - In a sample of 52 individuals receiving treatment, 40.4% had an immune response, which is not different from 30% (p-value = 0.102).

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Conditions for one proportion: test vs. CI

Confidence interval conditions

1. *Independent observations*
   - The observations were collected independently.

2. The number of successes and failures is at least 10:

$$n\hat{p} \geq 10, \quad n(1-\hat{p}) \geq 10$$

$\hat{p}$: sample prop for CI

Hypothesis test conditions

1. *Independent observations*
   - The observations were collected independently.

2. The number of **expected** successes and **expected** failures is at least 10.

$$np_0 \geq 10, \quad n(1-p_0) \geq 10$$

p_0 (null value) for hyp test

95% CI for population proportion

What to use for SE in CI formula?

$$\hat{p} \pm z^* \cdot SE_{\hat{p}}$$

Sampling distribution of $\hat{p}$:

$$\hat{p} \sim N\left(\mu_{\hat{p}} = p, \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}\right)$$

Problem: We don't know what p is - it's what we're estimating with the CI.
Solution: approximate p with $\hat{p}$:

$$SE_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- Note that I am not using a continuity correction here! This means our "by hand" calculation will be different than our R calculation
    - Using the continuity correction is more widely accepted
    - So I would suggest using R to calculate the confidence intervals when you can!

95% CI for population proportion of immune response by hand

95% CI for population ~~mean difference~~ proportion p:

$$\underline{\hat{p}} \pm \underline{z^*} \cdot \underline{SE}_{\hat{p}}$$

$$\hat{p} \pm z^* \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\underline{0.404} \pm \underline{1.96} \cdot \sqrt{\frac{0.404(1-0.404)}{52}}$$

$$0.404 \pm 1.96 \cdot 0.068$$

$$0.404 \pm 0.133$$

$$\underline{(0.27, 0.537)}$$

Used z^* = `qnorm(0.975)` = 1.96

“By hand” Conclusion:
We are 95% confident that the (population) proportion of individuals with an immune response is between 0.27 and 0.537.

95% CI for population proportion of immune response using R

- We can use R to get similar values

do NOT include p = 0.3

```r
prop.test(x = 21, n = 52, conf.level = 0.95, correct = T)
```

```
    1-sample proportions test with continuity correction

data:  21 out of 52, null probability 0.5
X-squared = 1.5577, df = 1, p-value = 0.212
alternative hypothesis: true p is not equal to 0.5
95 percent confidence interval:
 0.2731269 0.5487141
sample estimates:
        p 
0.4038462 
```

R Conclusion:

We are 95% confident that the (population) proportion of individuals with an immune response is between 0.273 and 0.549.

- Note: We expect some differences between the confidence interval calculated by hand vs. by R. R uses a slightly different method to calculate.

Break Time!

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Inference for difference of two independent proportions

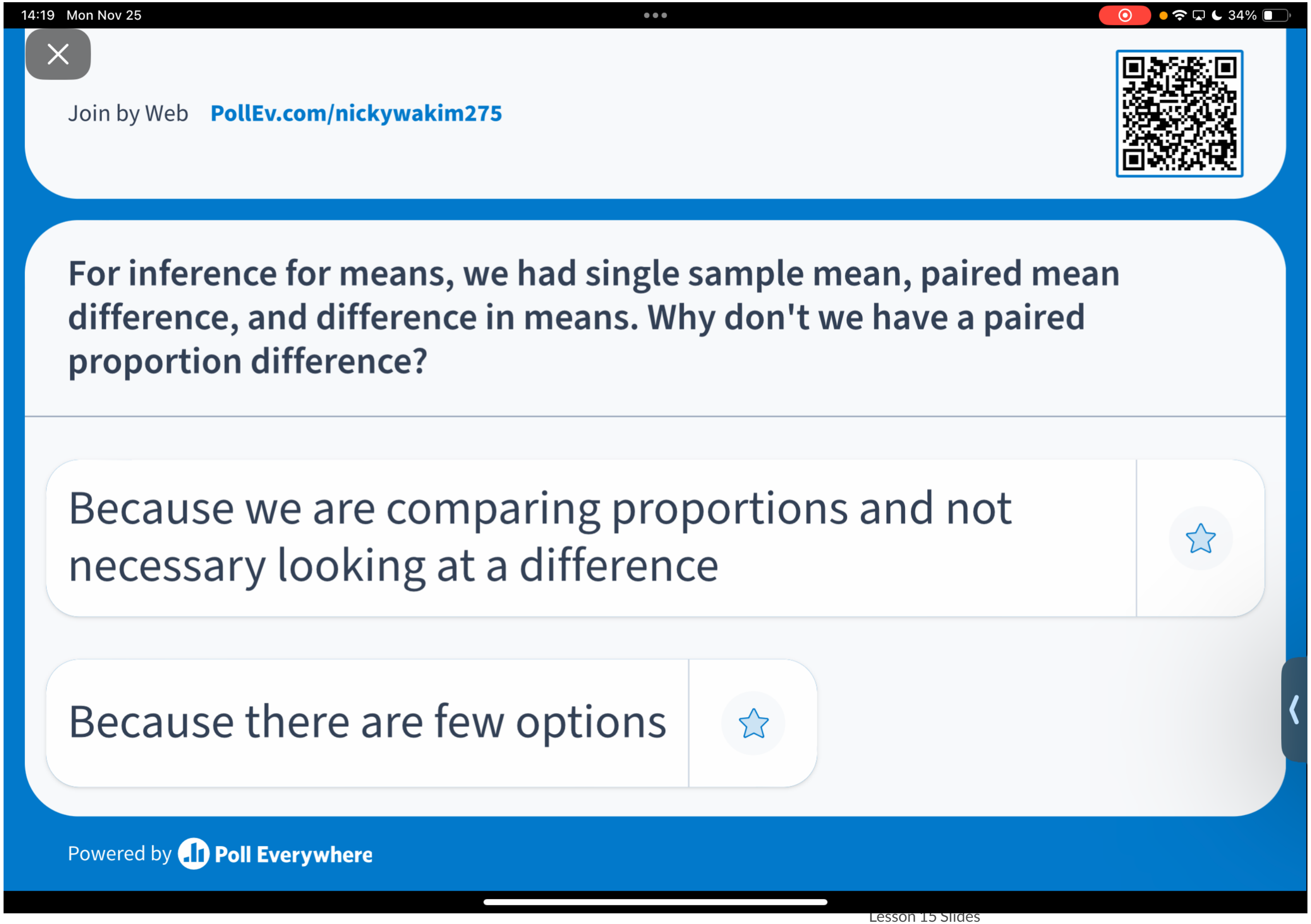
$\hat{p}_1 - \hat{p}_2$

- For means, we went from *inferences on single sample mean* to *inferences on difference in means from two independent samples*

- We can do the same thing for proportions

- We will go from *inferences on single sample proportion* to *inferences on difference in proportions from two independent samples*

Poll Everywhere Question 3



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For inference for means, we had single sample mean, paired mean difference, and difference in means. Why don't we have a paired proportion difference?

- Because we are comparing proportions and not necessary looking at a difference ☆
- Because there are few options ☆

ind 1

before after

132 145

diff ⑬

before after

→ No Yes

cannot take diff w/in pair

For difference in proportions: Population parameters vs. sample statistics

Population parameter

- Population 1 proportion: p_1, π_1 (“pi”)
- Population 2 proportion: p_2, π_2 (“pi”)

- Difference in proportions: $p_1 - p_2$

Sample statistic (point estimate)

- Sample 1 proportion: $\hat{p}_1, \hat{\pi}_1$ (“pi”)
- Sample 1 proportion: $\hat{p}_2, \hat{\pi}_2$ (“pi”)

- Difference in proportions: $\hat{p}_1 - \hat{p}_2$

very similar $\mu_1 - \mu_2$

Sampling distribution of $\hat{p}_1 - \hat{p}_2$

- $\hat{p}_1 = \frac{X_1}{n_1}$ and $\hat{p}_2 = \frac{X_2}{n_2}$,
    - X_1 & X_2 are the number of "successes"
    - n_1 & n_2 are the sample sizes of the 1st & 2nd samples

- Each $\hat{p}$ can be approximated by a normal distribution, for "big enough" n/p
- Since the difference of independent normal random variables is also normal, it follows that for "big enough" n_1 and n_2

diff of 2 normal dist'ns is Normal

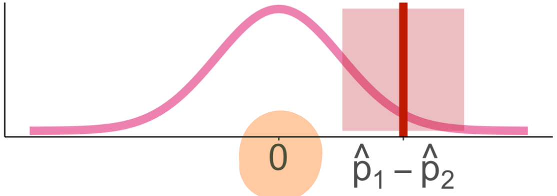
$$(\hat{p}_1 - \hat{p}_2 \sim N\left(\mu_{\hat{p}_1-\hat{p}_2} = p_1 - p_2, \sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{p_1 \cdot (1-p_1)}{n_1} + \frac{p_2 \cdot (1-p_2)}{n_2}}\right)$$

- What is "big enough"? At least 10 successes and 10 failures are expected in the sample: $n_1 p \geq 10$, $n_1(1-p) \geq 10$, $n_2 p \geq 10$, and $n_2(1-p) \geq 10$

Approaches to answer a research question

- **Research question is a generic form for a single proportion:** Is there evidence to support that the population proportions are different from each other?

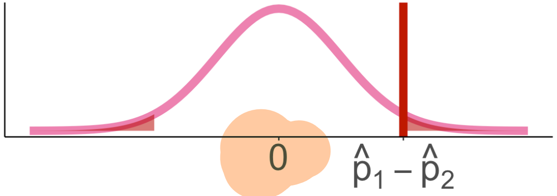
Calculate **CI for the proportion difference** $p_1 - p_2$:



$$\hat{p}_1 - \hat{p}_2 \pm z^* \cdot SE_{\hat{p}_1 - \hat{p}_2}$$

- with z^* = z-score that aligns with specific confidence interval

Run a **hypothesis test:**



Hypotheses

$$H_0 : p_1 - p_2 = 0$$
$$H_A : p_1 - p_2 \neq 0$$
$$(or \quad <, >)$$

Test statistic

$$z_{\hat{p}_1 - \hat{p}_2} = \frac{\hat{p}_1 - \hat{p}_2}{SE_{pool}}$$

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6. Construct and interpret confidence intervals for a difference in proportions.

Motivating example: effectiveness of mammograms

A 30-year study to investigate the effectiveness of mammograms versus a standard non-mammogram breast cancer exam was conducted in Canada with 89,835 participants. Each person was randomized to receive either annual mammograms or standard physical exams for breast cancer over a 5-year screening period.

By the end of the 25-year follow-up period, 1,005 people died from breast cancer. The results are summarized in the following table.

▶ Displaying the contingency table in R

	Death from breast cancer?		
Group	Yes	No	Total
Control Group	505	44405	44910
Mammogram Group	500	44425	44925
Total	1005	88830	89835

Reference: Steps in a Hypothesis Test

1. Check the **assumptions**
2. Set the **level of significance** α
3. Specify the **null** (H_0) and **alternative** (H_A) **hypotheses**
    1. In symbols
    2. In words
    3. Alternative: one- or two-sided?
4. Calculate the **test statistic**.
5. Calculate the **p-value** based on the observed test statistic and its sampling distribution
6. Write a **conclusion** to the hypothesis test
    1. Do we reject or fail to reject H_0?
    2. Write a conclusion in the context of the problem

Before we start, we need to calculate the pooled proportion ☆

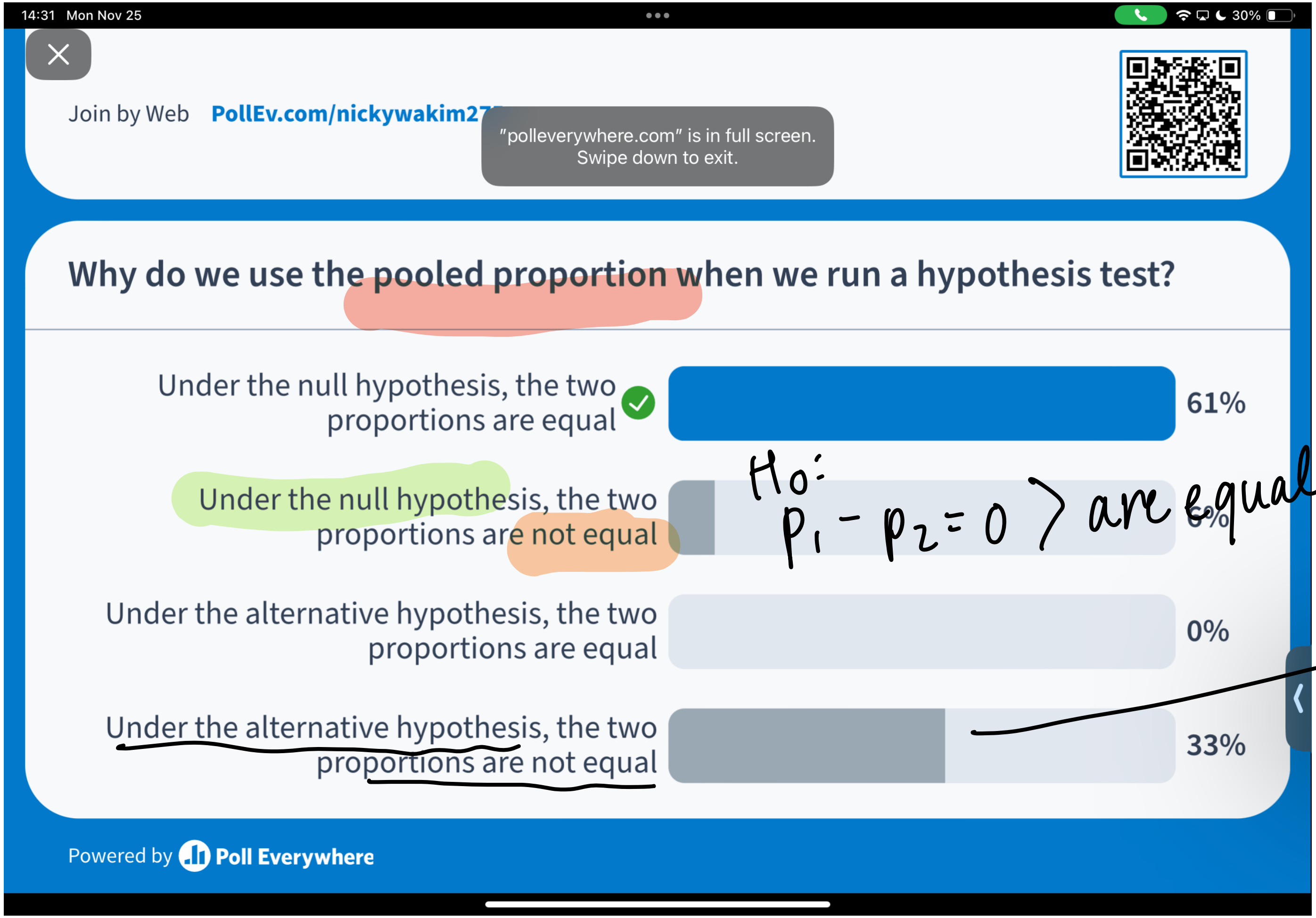
- Often, our null hypothesis is that the two proportions are equal
    - And that both populations are the same
- Thus, we calculate a pooled proportion to represent the proportion under the null distribution

$$\text{pooled proportion} = \hat{p}_{pool} = \frac{\text{total number of successes}}{\text{total number of cases}} = \frac{x_1 + x_2}{n_1 + n_2}$$

- In this example:

$$\hat{p}_{pool} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{500 + 505}{(500 + 44425) + (505 + 44405)} = 0.01119$$

Poll Everywhere Question 4



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"polleverywhere.com" is in full screen.
Swipe down to exit.

Why do we use the pooled proportion when we run a hypothesis test?

Under the null hypothesis, the two proportions are equal ✓	61%
Under the null hypothesis, the two proportions are not equal	6%
Under the alternative hypothesis, the two proportions are equal	0%
Under the alternative hypothesis, the two proportions are not equal	33%

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Ho:
$p_1 - p_2 = 0$ } are equal!

not why we are pooling prop's

Step 1: Check the assumptions

Conditions:

- *Independent observations & samples* ✓
  - The observations were collected independently. ✓
  - In particular, observations from the two groups weren't paired in any meaningful way.

samples are independent

- The number of expected successes and expected failures is at least 10 *for each group* - using the pooled proportion:
  - $n_1\hat{p}_{pool} \geq 10, \quad n_1(1-\hat{p}_{pool}) \geq 10$
  - $n_2\hat{p}_{pool} \geq 10, \quad n_2(1-\hat{p}_{pool}) \geq 10$

prev: $\hat{p}_{pool} = 0.01119$

- In the example, we check:
  - $n_1\hat{p}_{pool} = 44925 \cdot 0.0112 = 502.5839 \geq 10$ ✓
  - $n_1(1-\hat{p}_{pool}) = 44925(1-0.0112) = 44422.42 \geq 10$ ✓
  - $n_2\hat{p}_{pool} = 44910 \cdot 0.0112 = 502.4161 \geq 10$ ✓
  - $n_2(1-\hat{p}_{pool}) = 44910(1-0.0112) = 44407.58 \geq 10$ ✓

Step 3: Null and Alternative Hypothesis test

Two samples test

- H_0: The difference in population proportions of deaths from breast cancer among people who received annual mammograms and annual physical check-ups is 0.
- H_A: The difference in population proportions of deaths from breast cancer among people who received annual mammograms and annual physical check-ups is not 0.

$$H_0 : p_{mamm} - p_{ctrl} = 0$$
$$H_A : p_{mamm} - p_{ctrl} \neq 0$$

Step 4: Test statistic (1/2)

Sampling distribution of $\hat{p}_1 - \hat{p}_2$:

$p_2 = p_{pooled}$ $p_1 = p_{pooled}$

$$\hat{p}_1 - \hat{p}_2 \sim N\left(\mu_{\hat{p}_1-\hat{p}_2} = p_1 - p_2,\ \sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{p_1 \cdot (1-p_1)}{n_1} + \frac{p_2 \cdot (1-p_2)}{n_2}}\right)$$

replace w/ pooled

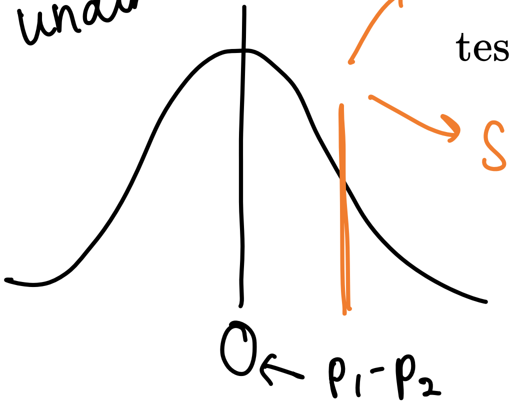
Since we assume $H_0: p_1 - p_2 = 0$ is true, we "pool" the proportions of the two samples to calculate the SE:

$$\text{pooled proportion} = \hat{p}_{pool} = \frac{\text{total number of successes}}{\text{total number of cases}} = \frac{x_1 + x_2}{n_1 + n_2}$$

b/c $p_1 = p_2 = p_{pooled}$

Test statistic:

under null



$$\text{test statistic} = z_{\hat{p}_1-\hat{p}_2} = \frac{\hat{p}_1 - \hat{p}_2 - 0}{\sqrt{\frac{\hat{p}_{pool}(1-\hat{p}_{pool})}{n_1} + \frac{\hat{p}_{pool}(1-\hat{p}_{pool})}{n_2}}}$$

SE =

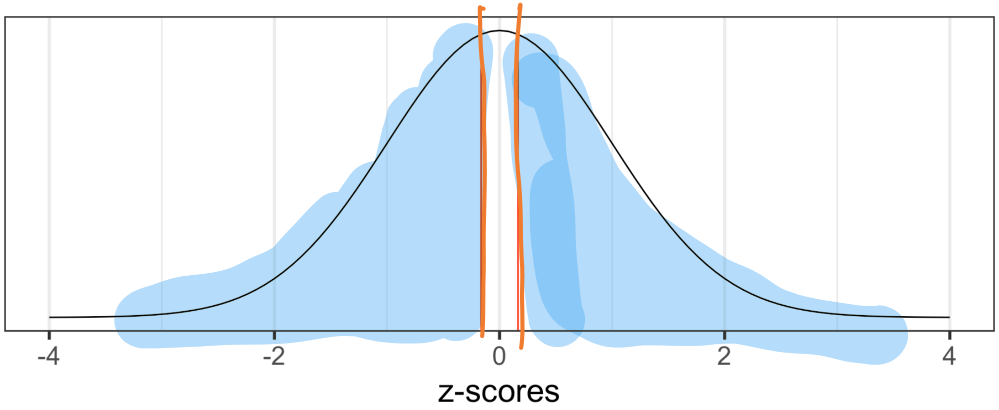
Step 4: Test statistic (2/2)

From our example: Recall that $\hat{p}_1 = \frac{500}{44925} = 0.0111$, $\hat{p}_2 = \frac{505}{44910} = 0.0112$, $n_1 = 44925$, $n_2 = 44910$, and $\hat{p}_{pool} = 0.01119$

The test statistic is:

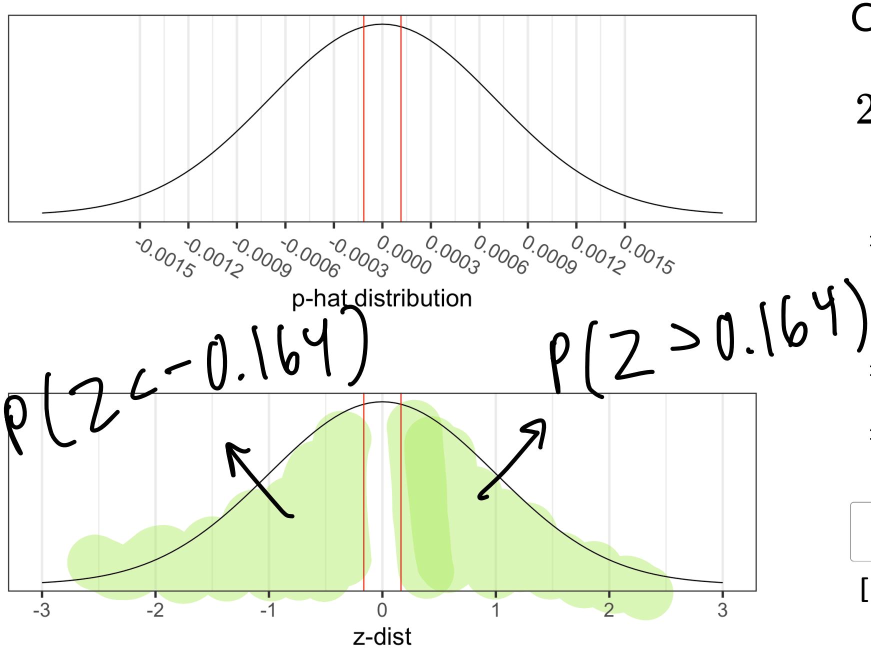
$$z_{\hat{p}_1 - \hat{p}_2} = \frac{\hat{p}_1 - \hat{p}_2 - 0}{\sqrt{\frac{\hat{p}_{pool}\cdot(1-\hat{p}_{pool})}{n_1} + \frac{\hat{p}_{pool}\cdot(1-\hat{p}_{pool})}{n_2}}} = \frac{0.0111 - 0.0112}{\sqrt{\frac{0.01119\cdot(1-0.01119)}{44925} + \frac{0.01119\cdot(1-0.01119)}{44910}}} = -0.163933$$

- Let's see the z-score on a Z-distribution (Standard Normal curve)



Step 5: p-value

The **p-value** is the **probability** of obtaining a test statistic *just as extreme or more extreme* than the observed test statistic assuming the null hypothesis H_0 is true.



Calculate the *p*-value:

$$2 \cdot P(\hat{p}_1 - \hat{p}_2 < 0.0111 - 0.0112)$$

$$= P\left(Z_{\hat{p}_1-\hat{p}_2} < \frac{0.0111 - 0.0112}{\sqrt{\frac{0.01119\cdot(1-0.01119)}{44925} + \frac{0.01119\cdot(1-0.01119)}{44910}}}\right)$$

$$= \underline{2 \cdot P(Z_{\hat{p}} < -0.164)}$$

$$= \underline{0.8697839}$$

```
2*pnorm(-0.1639)
```

```
[1] 0.8698099
```

Step 4-5: test statistic and p-value together using `prop.test()`

```r
prop.test(x = c(505, 500), n = c(44910, 44925)) # no p needed
```

mamm ctrl mamm ctrl

```
    2-sample test for equality of proportions with continuity correction

data:  c(505, 500) out of c(44910, 44925)
X-squared = 0.01748, df = 1, p-value = 0.8948
alternative hypothesis: two.sided
95 percent confidence interval:
 -0.001282751  0.001512853
sample estimates:
    prop 1     prop 2
0.01124471 0.01112966
```

▶ Tidying the output of `prop.test()`

estimate1	estimate2	statistic	p.value	parameter	conf.low	conf.high	method	alternative
0.01124471	0.01112966	0.01747975	0.8948174	1	-0.001282751	0.001512853	2-sample test for equality of proportions with continuity correction	two.sided

- Note: We expect some differences between the test statistic and p-value calculated by hand vs. by R. R uses a slightly different method to calculate.

Step 6: Conclusion to hypothesis test

$$H_0 : p_{mamm} - p_{ctrl} = 0$$
$$H_A : p_{mamm} - p_{ctrl} \neq 0$$

$$p\text{-val} = 0.87 > 0.05$$

- Recall the p-value = 0.8698
- Use α = 0.05
- Do we reject or fail to reject H_0?

Conclusion statement:

- Stats class conclusion
  - There is insufficient evidence that the difference in (population) proportions of deaths from breast cancer among people who received annual mammograms and annual physical check-ups are different (p-value = 0.87).
- More realistic manuscript conclusion:

  $\hat{p}_2$

  - 1.11% of people receiving annual mammograms (n=44925) and 1.12% of people receiving annual physical exams (n=44925) died from breast cancer (p-value = 0.87).

  $\hat{p}_1$

Learning Objectives

1. Remind ourselves of the Normal approximation of the binomial distribution and define the sampling distribution of a sample proportion
2. Run a hypothesis test for a single proportion and interpret the results.
3. Construct and interpret confidence intervals for a single proportion.
4. Understand how CLT applies to a difference in binomial random variables
5. Run a hypothesis test for a difference in proportions and interpret the results.
6. Construct and interpret confidence intervals for a difference in proportions.

Conditions for difference in proportions: test vs. CI

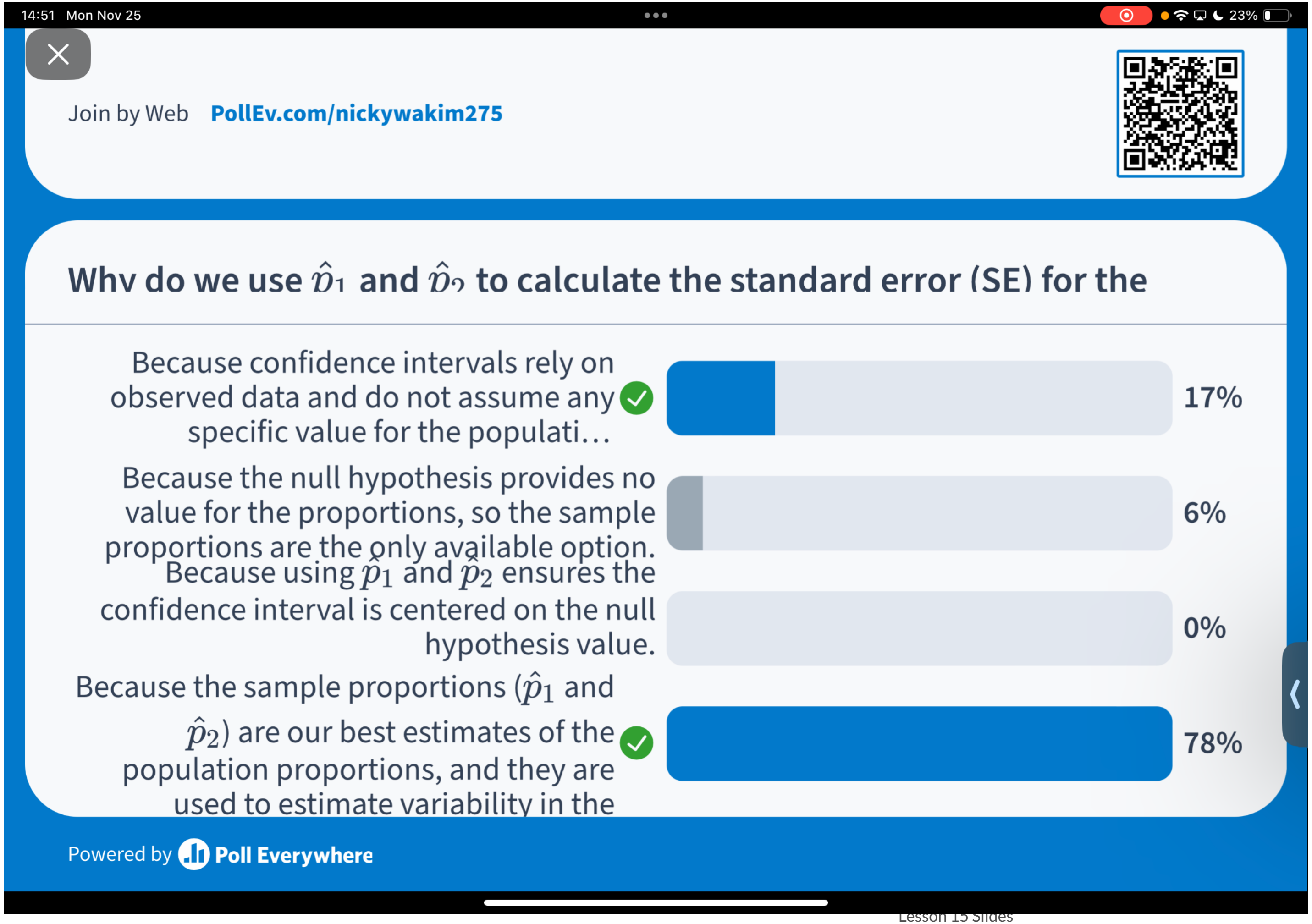
Confidence interval conditions

1. *Independent observations & samples*
   - The observations were collected independently.
   - In particular, observations from the two groups weren't paired in any meaningful way.
2. The number of successes and failures is at least 10 *for each group.*
   - $n_1\hat{p}_1 \geq 10, \quad n_1(1-\hat{p}_1) \geq 10$
   - $n_2\hat{p}_2 \geq 10, \quad n_2(1-\hat{p}_2) \geq 10$

Hypothesis test conditions

1. *Independent observations & samples*
   - The observations were collected independently.
   - In particular, observations from the two groups weren't paired in any meaningful way.
2. The number of **expected** successes and **expected** failures is at least 10 *for each group* - using the pooled proportion:
   - $n_1\hat{p}_{pool} \geq 10, \quad n_1(1-\hat{p}_{pool}) \geq 10$
   - $n_2\hat{p}_{pool} \geq 10, \quad n_2(1-\hat{p}_{pool}) \geq 10$

Poll Everywhere Question 5



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Why do we use $\hat{p}_1$ and $\hat{p}_2$ to calculate the standard error (SE) for the

Because confidence intervals rely on observed data and do not assume any specific value for the populati…	✓	17%
Because the null hypothesis provides no value for the proportions, so the sample proportions are the only available option.		6%
Because using $\hat{p}_1$ and $\hat{p}_2$ ensures the confidence interval is centered on the null hypothesis value.		0%
Because the sample proportions ($\hat{p}_1$ and $\hat{p}_2$) are our best estimates of the population proportions, and they are used to estimate variability in the	✓	78%

95% CI for population difference in proportions

What to use for SE in CI formula?

$$\hat{p}_1 - \hat{p}_2 \pm z^* \cdot SE_{\hat{p}_1 - \hat{p}_2}$$

SE in sampling distribution of $\hat{p}_1 - \hat{p}_2$

$$\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{p_1 \cdot (1 - p_1)}{n_1} + \frac{p_2 \cdot (1 - p_2)}{n_2}}$$

Problem: We don't know what p is - it's what we're estimating with the CI.
Solution: approximate p_1, p_2 with $\hat{p}_1, \hat{p}_2$:

★ $$SE_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{\underline{\hat{p}_1} \cdot (1 - \underline{\hat{p}_1})}{n_1} + \frac{\underline{\hat{p}_2} \cdot (1 - \underline{\hat{p}_2})}{n_2}}$$

no longer pooled

95% CI for the population difference in proportions

95% CI for population mean difference $p_1 - p_2$:

$$\underline{\hat{p}_1 - \hat{p}_2} \pm \underline{z^*} \cdot \underline{SE_{\hat{p}_1 - \hat{p}_2}}$$

$$\hat{p}_1 - \hat{p}_2 \pm z^* \cdot \sqrt{\frac{\hat{p}_1 \cdot (1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2 \cdot (1 - \hat{p}_2)}{n_2}}$$

$$0.01113 - 0.01124 \pm 1.96 \cdot \sqrt{\frac{0.01113 \cdot (1 - 0.01113)}{44925} + \frac{0.01124 \cdot (1 - 0.01124)}{44910}}$$

$$0.35 \pm 1.96 \cdot 0.001$$

$$0.35 \pm 0.002$$

$$(-0.002, 0.002) \quad ✓$$

Used z^* = `qnorm(0.975)` = 1.96

Interpretation:
We are 95% confident that the difference in (population) proportions of deaths due to breast cancer comparing people who received annual mammograms to annual physical check-ups is between -0.002 and 0.002.

95% CI for the population difference in proportions

- We can use R to get similar values

```r
prop.test(x = c(505, 500), n = c(44910, 44925))
```

mamm ctrl mamm ctrl

```
    2-sample test for equality of proportions with continuity correction

data:  c(505, 500) out of c(44910, 44925)
X-squared = 0.01748, df = 1, p-value = 0.8948
alternative hypothesis: two.sided
95 percent confidence interval:
 -0.001282751  0.001512853
sample estimates:
    prop 1     prop 2 
0.01124471 0.01112966 
```

R Conclusion:

We are 95% confident that the difference in (population) proportions of deaths due to breast cancer comparing people who received annual mammograms to annual physical check-ups is between -0.0013 and 0.0015.

- Note: We expect some differences between the confidence interval calculated by hand vs. by R. R uses a slightly different method to calculate.