Lesson 16: Poisson Regression

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Dumbo Octopus

Learning Objectives

1. Review Generalized Linear Models and how we can branch to other types of regression.
2. Understand what we can measure with Poisson regression and how to interpret coefficients.
3. Understand how to adjust for different follow-up times among individuals

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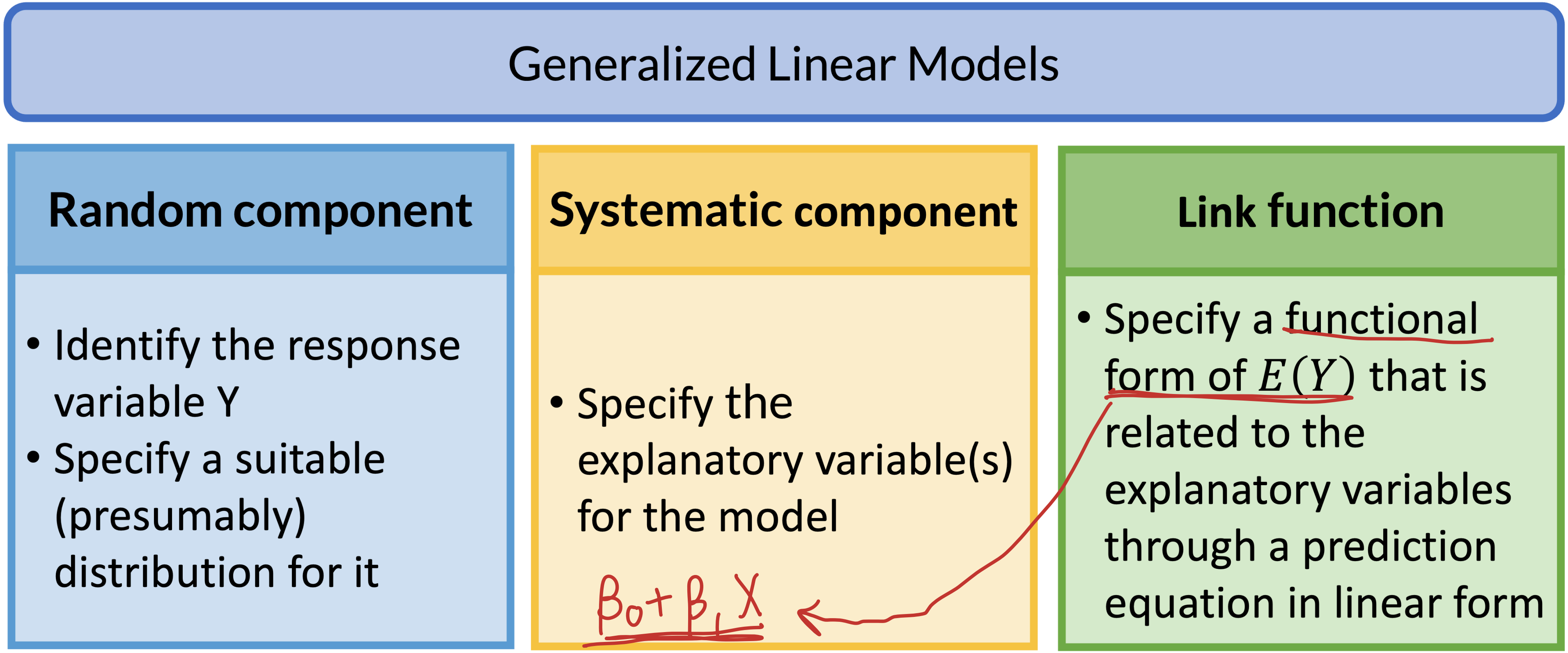
Review: Generalized Linear Models (GLMs)



Generalized Linear Models

Random component

- Identify the response variable Y
- Specify a suitable (presumably) distribution for it

Systematic component

- Specify the explanatory variable(s) for the model

$\underline{\beta_0+\beta_1 X}$

Link function

- Specify a functional form of $E(Y)$ that is related to the explanatory variables through a prediction equation in linear form

GLM: Random Component

- The random component specifies the response variable Y and selects a probability distribution for it

- Basically, we are just identifying the distribution for our outcome
  - If Y is binary: assumes a binomial distribution of Y
  - If Y is count: assumes Poisson or negative binomial distribution of Y
  - If Y is continuous: assumea Normal distribution of Y

GLM: Systematic Component

- The systematic component specifies the explanatory variables, which enter linearly as predictors

$$\underline{\beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k}$$

- Above equation includes:
  - Centered variables
  - Interactions
  - Transformations of variables (like squares)

- Systematic component is the **same** as what we learned in Linear Models

GLM: Link Function

- If $\mu = E(Y)$, then the link function specifies a function $g(.)$ that relates μ to the linear predictor as:

$$\text{fn}\;\boxed{g\left(\overset{E(Y)}{\mu}\right)} = \underbrace{\beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k}_{\text{sys}}$$

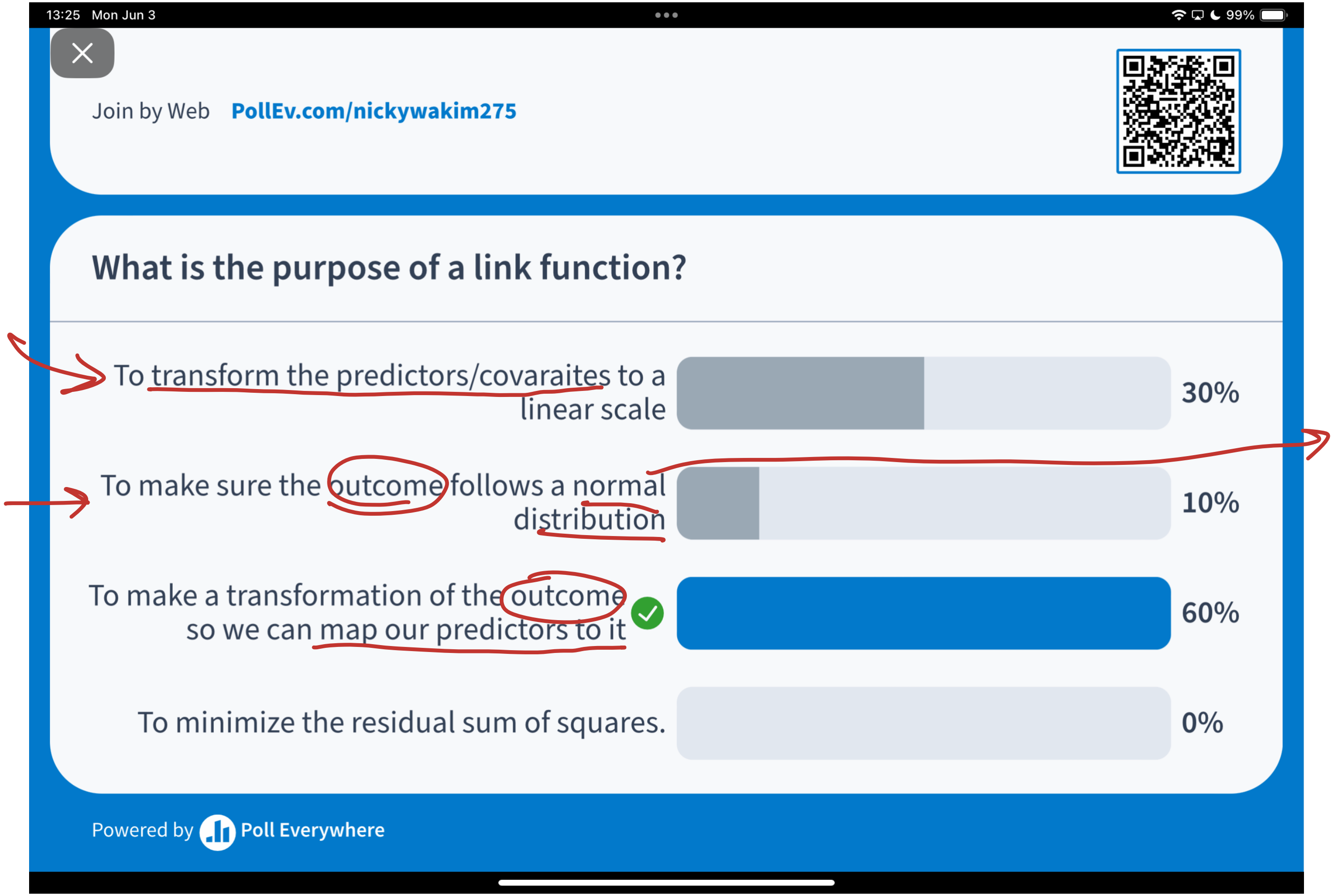
  - $g(\mu)$ is the transformation we make to $E(Y)$ (aka μ) so that the linear predictors (right side of equation) can be linked to the outcome
- The link function connects the random component with the systematic component
- Can also think of this as:

$$\mu = g^{-1}\left(\beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k\right)$$

GLM: Link Function

Link	Link function	Type of response variable	Type of regression
normally dist y Identity link	$g(\mu) = 1\times\mu$	Continuous response variables	Linear regression
count Log link	$g(\mu) = \log(\mu)$	Discrete count response variable	Poisson regression
binary Logit link	$g(\mu) = \text{logit}(\mu)$ $= \log\left[\frac{\mu}{1-\mu}\right]$	Categorical response variable	Logistic regression
binary Log link	$g(\mu) = \log(\mu)$	Categorical response variable	Log-binomial regression

Poll Everywhere Question 1



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What is the purpose of a link function?

To transform the predictors/covaraites to a linear scale 30%

To make sure the outcome follows a normal distribution 10%

To make a transformation of the outcome so we can map our predictors to it ✓ 60%

To minimize the residual sum of squares. 0%

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transformation closer to Normal distribution

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Poisson Distribution

- This distribution is often used to model count data
- Examples:
  - Distribution of number of deaths due to lung cancer
  - Distribution of number of individuals diagnosed with leukemia
  - Distribution of number of hospitalizations

- This distribution is often used to model rate data

time becomes big component

- Examples:
  - Distribution of number of deaths due to lung cancer **per year**
  - Distribution of number of individuals diagnosed with leukemia **over follow-up time**
  - Distribution of number of hospitalizations **~~in~~ a day**

per

Example 1: Horseshoe Crabs and Satellites

Example of count data:

Each female horseshoe crab in the study had a male crab attached to her in her nest. The study investigated factors that affect whether the female crab had any other males, called satellites, residing near her. Explanatory variables that are thought to affect this included the female crab's color, spine condition, and carapace width, and weight. The response outcome for each female crab is the number of satellites. There are 173 females in this study.

```r
library(rsq)
data(hcrabs)
```

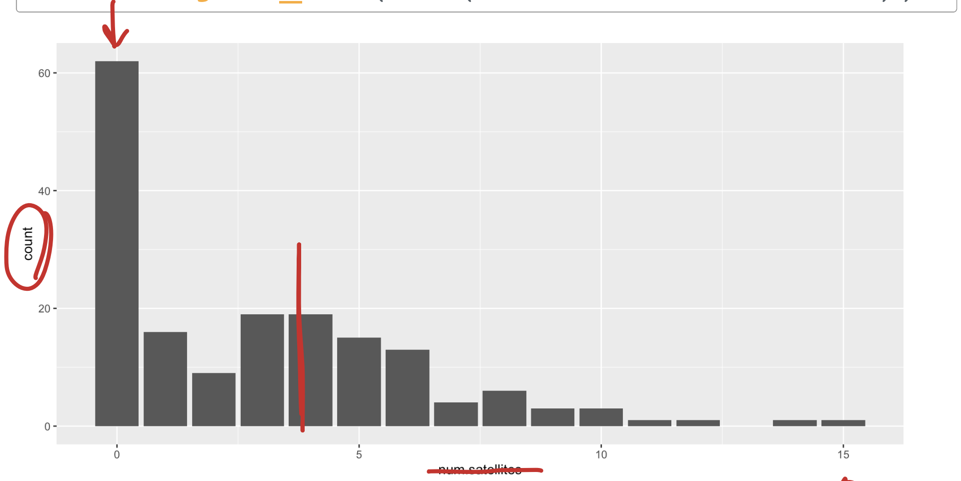
Poisson Distribution with a count

- The probability function of Poisson distribution:

$$P(Y = y|\overset{\beta_0+\beta_1 X}{\mu}) = \frac{\mu^y e^{-\mu}}{y!}$$

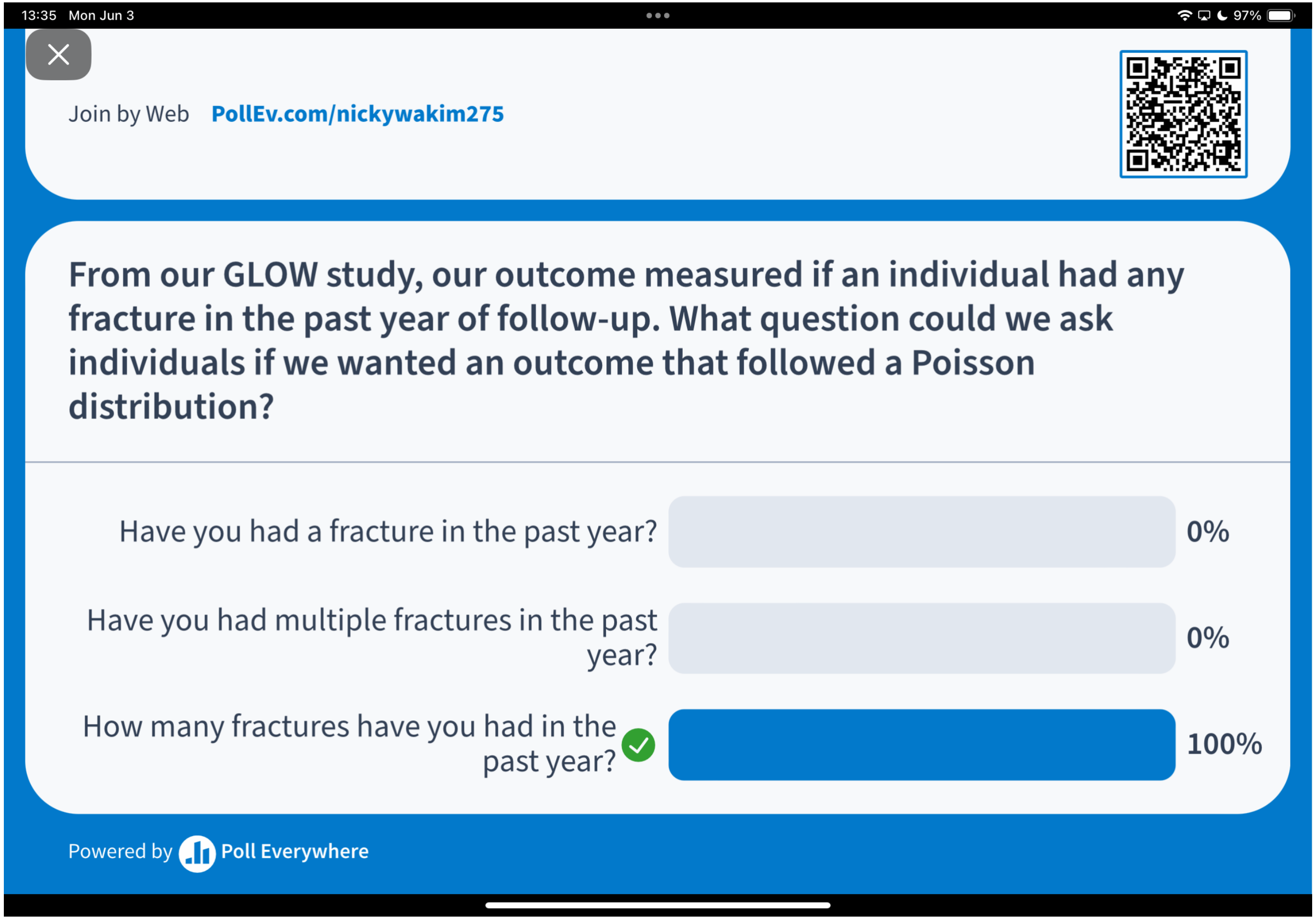
  - Where y's are non-negative integers $y = 0, 1, 2, \ldots$
  - Where μ is the mean of Y, that is $E(Y) = \mu$
  - And also, $\text{var}(Y) = \mu$
- For a Poisson distribution, $Y \sim \text{Poisson}(\mu)$
  - Range: $[0, \infty)$

```
ggplot(data = hcrabs) +
     geom_bar(aes(x = num.satellites))
```



satellites

Poll Everywhere Question 2



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From our GLOW study, our outcome measured if an individual had any fracture in the past year of follow-up. What question could we ask individuals if we wanted an outcome that followed a Poisson distribution?

Have you had a fracture in the past year?	0%
Have you had multiple fractures in the past year?	0%
How many fractures have you had in the past year? ✓	100%

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Example 2: Lung Cancer Incidence

Example of rate data:

We can look at the lung cancer incident counts (cases) per age group for four Danish cities from 1968 to 1971. Since it's reasonable to assume that the expected count of lung cancer incidents is proportional to the population size, we would prefer to model the rate of incidents per capita.

```r
library(ISwR)
data(eba1977)
lc_inc = eba1977 %>% mutate(lpop = log(pop))
```

Poisson Distribution with a rate

or in a population

- If we look at the probability of y events **in a time period** t for a Poisson random variable, we could write:

$$P(Y = y|\mu) = \frac{\mu^y e^{-\mu}}{y!} = \frac{(\lambda t)^y e^{-\lambda t}}{y!}$$

λt : rate × time

λt : rate × pop level

  - Where y's are non-negative integers $y = 0, 1, 2, \ldots$
  - Where $\mu = \lambda t$, where λ is the expected number of events per unit time (aka **rate**)
  - Then μ is the expected number of events over time t

event per 1000 ppl

What does λ represent in the Poisson distribution?

- What does λ represent?
  - A rate, the expected number of events in a given population over a given period time

- Example: Number of patient arrivals into the Emergency Room per hour

- The Poisson distribution is the prototype for assigning probabilities of observing any number of events

Review: Simple Logistic Regression

- Let Y is the dependent variable of interest and x is a predictor variable,
  - In simple logistic regression, we have

$$\overset{\text{logit link}}{\boxed{\log}}\left(\frac{\pi(X)}{1-\pi(X)}\right) = \underbrace{\beta_0 + \beta_1 X}_{\text{sys}}$$

  - where $\pi(X) = P(\underline{Y = 1} \mid X = x)$

Simple Poisson Regression Model

- What do we model in a Poisson regression?

- Log of conditional mean of Y given x
  - Conditional mean of Y given x is represented as $E(Y \mid X) = \mu(X)$
  - Let Y be a Poisson count **for a given unit of time**, then $\mu(X) = \lambda(X)$
  - In a simple Poisson regression, we have

log link

$$\ln(\mu(X)) = \ln(\lambda(X)) = \underbrace{\beta_0 + \beta_1 X}_{\text{sys}}$$

- So this is also called a **log-linear model**

for count:

Poisson regression

tells us link & random

binary 0, 1

log-binomial regression

↳ link & random

log

binary, bernoulli

Parameter Interpretation: Binary X (1/2)

- In simple Poisson regression:

$$\ln(\mu(X)) = \ln(\lambda(X)) = \underline{\beta_0 + \beta_1 X}$$

$X=1$
or $X=0$

- When X is a binary variable: How do we interpret β_1?
  - When $X = 0$:

$$\ln(\mu(X=0)) = \beta_0 + \beta_1 \cdot 0 = \beta_0$$

$\beta_0 = \log(\mu(X=0))$

$e^{\beta_0} = \mu(X=0)$

    - $\mu(X=0) = \exp(\beta_0)$: the mean count or rate of Y when $X = 0$
  - When $X = 1$:

$$\ln(\mu(X=1)) = \beta_0 + \beta_1 \cdot 1 = \beta_0 + \beta_1$$

    - $\mu(X=1) = \exp(\beta_0 + \beta_1)$: the mean count or rate of Y when $X = 1$

Parameter Interpretation: Binary X (2/2)

- When X is a binary variable: How do we interpret β_1?
  - By subtraction, we have

$$\log(a) - \log(b) = \log\left(\frac{a}{b}\right)$$

$$\beta_1 = \underbrace{\ln(\mu(X=1))}_{\beta_0+\beta_1} - \underbrace{\ln(\mu(X=0))}_{\beta_0} = \ln\left(\frac{\mu(X=1)}{\mu(X=0)}\right)$$

  - β_1: log-count ratio or log-rate ratio comparing $X=1$ to $X=0$
    - Since $\mu(X)$ is $\lambda(X)$ is the rate of Y
  - So $\exp(\beta_1)$ is the count or rate ratio!

in lin reg

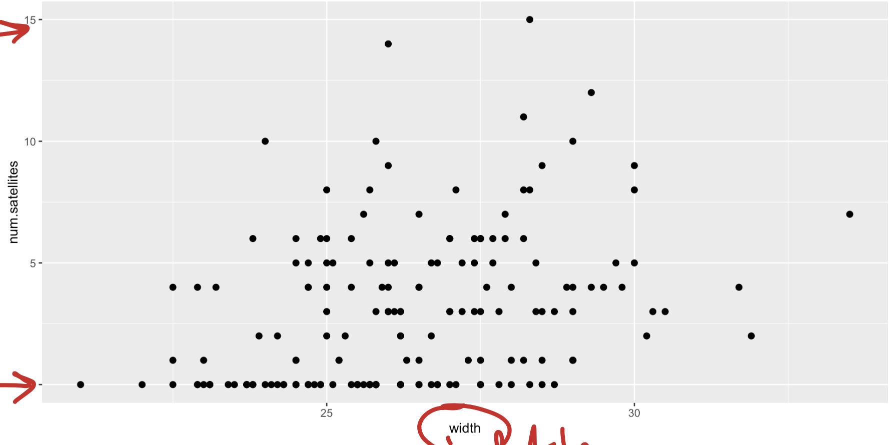
$$\beta_1 = E(Y \mid X=1) - E(Y \mid X=0)$$

Parameter Interpretation: Continuous X

- When X is a continuous variable: How do we interpret β_0?
  - β_0: log-count or log-rate when X is 0
  - So $\exp(\beta_0)$ is the expected count or rate when X is 0
- When X is a continuous variable: How do we interpret β_1?
  - β_1: log-count ratio or log-rate ratio for every 1 unit increase in X
    - Since $\mu(X)$ is $\lambda(X)$ is the rate of Y
  - So $\exp(\beta_1)$ is the rate ratio for every 1 unit increase in X

Example 1: Horseshoe Crabs and Satellites

```r
ggplot(hcrabs,
       aes(x=width,
           y=num.satellites)) +
  geom_point(size = 2)
```



sat

width

```r
crab_mod = glm(num.satellites ~ width,
               family=poisson,
               data=hcrabs)
tidy(crab_mod, conf.int=T,
     exponentiate=T) %>%
  gt() %>%
  tab_options(table.font.size = 35) %>%
  fmt_number(decimals = 2)
```

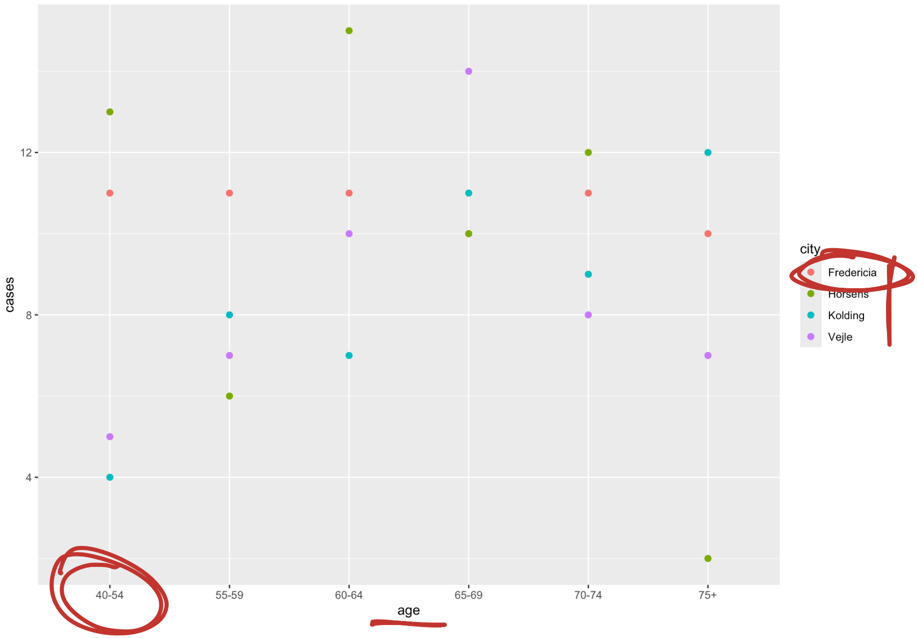
term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	0.04	0.54	−6.09	0.00	0.01	0.11
width	1.18	0.02	8.22	0.00	1.13	1.23

estimated count ratio: $\exp(\hat{\beta}_1)$

Interpretation: For every 1-cm increase in carapace width, the expected number of satellites increases by 18% (95% CI: 13%, 23%).

Example 2: Lung Cancer Incidence

```
ggplot(lc_inc, aes(x=age, y=cases,
                   color = city)) +
  geom_point(size = 2)
```



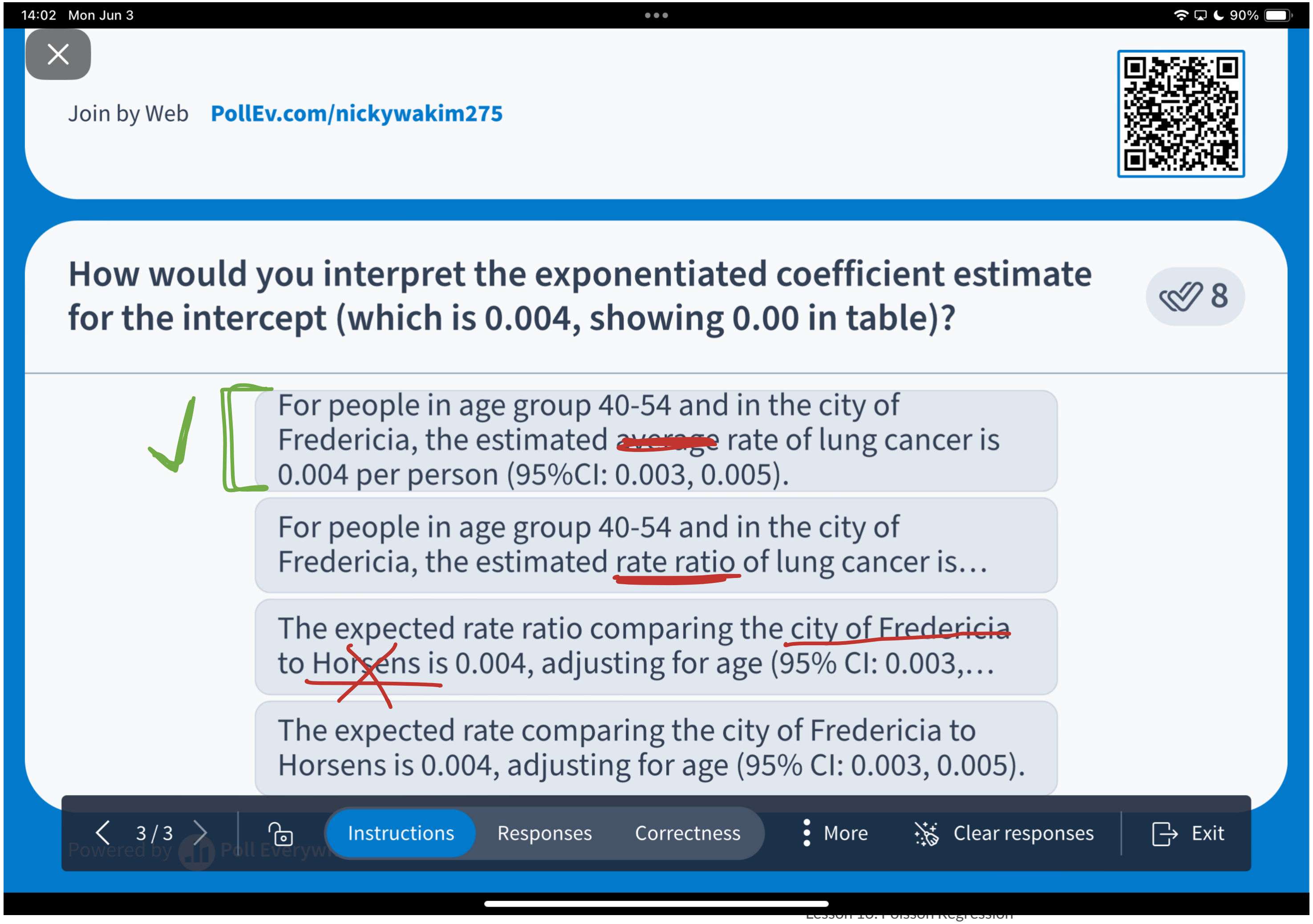
```
lc_mod = glm(cases ~ city + age,
             offset=lpop,
             family=poisson,
             data=lc_inc)
```

log pop
(per cap)

- Regression table

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	0.004	0.200	−28.125	0.000	0.002	0.005
cityHorsens	0.719	0.182	−1.818	0.069	0.503	1.026
cityKolding	0.690	0.188	−1.978	0.048	0.476	0.995
cityVejle	0.762	0.188	−1.450	0.147	0.525	1.099
age55-59	3.007	0.248	4.434	0.000	1.843	4.901
age60-64	4.566	0.232	6.556	0.000	2.907	7.236
age65-69	5.857	0.229	7.704	0.000	3.748	9.249
age70-74	6.404	0.235	7.891	0.000	4.043	10.212
age75+	4.136	0.250	5.672	0.000	2.523	6.762

Poll Everywhere Question 3



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How would you interpret the exponentiated coefficient estimate for the intercept (which is 0.004, showing 0.00 in table)?

✓ For people in age group 40-54 and in the city of Fredericia, the estimated ~~average~~ rate of lung cancer is 0.004 per person (95%CI: 0.003, 0.005).

For people in age group 40-54 and in the city of Fredericia, the estimated rate ratio of lung cancer is...

The expected rate ratio comparing the city of Fredericia to Horsens is 0.004, adjusting for age (95% CI: 0.003,...

The expected rate comparing the city of Fredericia to Horsens is 0.004, adjusting for age (95% CI: 0.003, 0.005).

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Building towards person-years

- In an example of number of patient arrivals, an event does not conclude the study
  - If someone arrives within the first minute of the study, then we keep counting
  - We may be able to study the association of arrivals with qualities of the hospital, but we can't measure qualities of the individuals arriving

- For example, in the lung cancer study, we can only discuss the incidence of cancer at the city-wide level

- What happens if we want to measure qualities of the individual?
  - We can measure a hospitalization rate

Why Person-Years?

- If we are measuring at the individual level and counting something that is “terminal” then our count will always be 0 or 1
  - Example: Number of individuals diagnosed with leukemia
  - This only happens once, so how do we measure the rate here?

- Since rate involves the counts and time – we can use the time to diagnosis to estimate the rate
  - Often expressed in units such as events per thousand person-years

- Person-years allow us to follow individuals for different amounts of time

What is a Person-Year and how to calculate?

- One person-year is a unit of time defined as one person being followed for one year

- Person-years for a sample of n subjects is calculated as the total years followed for the n subjects, where each subject could have different follow-up time

- Example: suppose we have 5 subjects, two of the subjects were followed for 2 years, and two of them are followed for 3 years and the fifth subject was followed for 3.8 years

$$\text{person-years} = 2 \text{ people} \cdot 2 \text{ years} + 2 \text{ people} \cdot 3 \text{ years} + 1 \text{ person} \cdot 3.8 \text{ years} = 13.8 \text{ person-years}$$

Calculating Rate

- Suppose that we observe one event during the follow-up period, then

$$\begin{aligned}\text{Rate of event} &= \frac{\#events}{\text{person-years}} = \frac{1 \text{ event}}{13.8 \text{person-years}} \\ &= 0.072 \text{ events per person}-\text{year} \\ &= 72 \text{ events per } 1000 \text{ person}-\text{years}\end{aligned}$$

- Now our rate of event is measured per person-year
  - Often use 1000 person-years to count the events with whole numbers

Including an offset when we have different follow-up times

- What if we have data that each observation has different period of time?
  - For example, we look at number of ED visits when subjects are enrolled in Oregon Health Plan, however, each subjects are enrolled in the plan for different length of time…
  - How do we incorporate this different length of time?
- Note we have: $\mu = \lambda t$ and with predictor X, $\mu(X) = \lambda(X) \cdot t(X)$
  - t is now a function of the individual, represented with X
- Then we construct:

$$\begin{aligned} \ln(\lambda(X)) &= \beta_0 + \beta_1 X \\ \ln(\lambda(X)) = \ln\left(\frac{\mu(X)}{t(X)}\right) = \ln(\mu(X)) - \ln(t(X)) &= \beta_0 + \beta_1 X \\ \ln(\mu(X)) &= \ln(t(X)) + \beta_0 + \beta_1 X \end{aligned}$$

Including an offset when we have different follow-up times

- That is, to incorporate the different lengths in the model,

$$\ln(\mu(X)) = \ln(t(X)) + \beta_0 + \beta_1 X$$

- We have one more term in the model and this term is called **offset**, a known term in the model since $t(X)$ is known for each individual
- $\ln(t(X))$ is called the offset
- Offsets can also be something like the population size in a city…

Example 2: Lung Cancer Incidence

```r
lc_mod = glm(cases ~ city + age, offset=lpop, family=poisson, data=lc_inc)
summary(lc_mod)
```

```
Call:
glm(formula = cases ~ city + age, family = poisson, data = lc_inc,
    offset = lpop)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)    
(Intercept)  -5.6321     0.2003 -28.125  < 2e-16 ***
cityHorsens  -0.3301     0.1815  -1.818   0.0690 .  
cityKolding  -0.3715     0.1878  -1.978   0.0479 *  
cityVejle    -0.2723     0.1879  -1.450   0.1472    
age55-59      1.1010     0.2483   4.434 9.23e-06 ***
age60-64      1.5186     0.2316   6.556 5.53e-11 ***
age65-69      1.7677     0.2294   7.704 1.31e-14 ***
age70-74      1.8569     0.2353   7.891 3.00e-15 ***
age75+        1.4197     0.2503   5.672 1.41e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 129.908  on 23  degrees of freedom
Residual deviance:  23.447  on 15  degrees of freedom
AIC: 137.84
```

```
Number of Fisher Scoring iterations: 5
```

Further reading / tutorials on Poisson regression

- Good tutorial in R
- When people are followed for different amounts of time, we should include an offset
  - Poisson Regression Modeling Using Rate Data: section from above site that discusses offsets
- We can use Wald test and LRT in the same way as logistic regression to test our coefficients and variables