SLR: Inference and Prediction

Nicky Wakim

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Learning Objectives

1. Estimate the variance of the residuals
2. Using a hypothesis test, determine if there is enough evidence that population slope β_1 is not 0 (applies to β_0 as well)
3. Calculate and report the estimate and confidence interval for the population slope β_1 (applies to β_0 as well)
4. Calculate and report the estimate and confidence interval for the expected/mean response given X

Let's revisit the regression analysis process

Model Selection

- Building a model
- Selecting variables
- Prediction vs interpretation
- Comparing potential models

Model Fitting

- Find best fit line
- Using OLS in this class
- Parameter estimation
- Categorical covariates
- Interactions

Model Evaluation

- Evaluation of model fit
- Testing model assumptions
- Residuals
- Transformations
- Influential points
- Multicollinearity

Model Use (Inference)

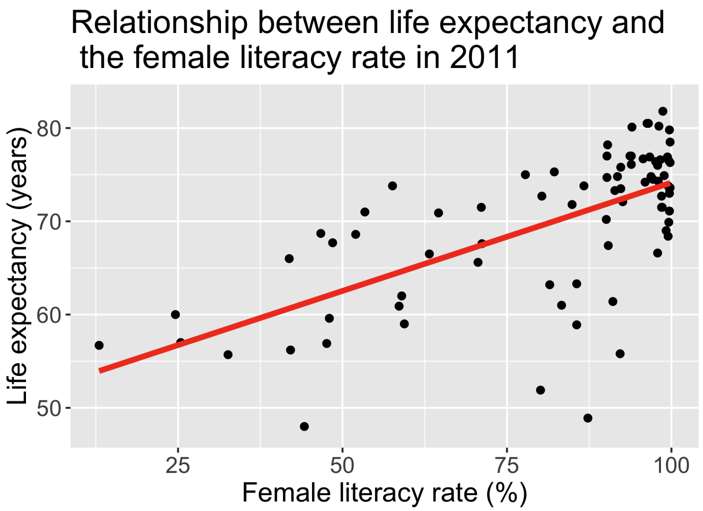
- Inference for coefficients
- Hypothesis testing for coefficients
- Inference for expected Y given X
- Prediction of new Y given X

Let's remind ourselves of the model that we fit last lesson

- We fit Gapminder data with female literacy rate as our independent variable and life expectancy as our dependent variable
- We used OLS to find the coefficient estimates of our best-fit line

```
model1 <- lm(life_expectancy_years_2011 ~
                 female_literacy_rate_2011,
                   data = gapm)
# Get regression table:
tidy(model1) %>% gt() %>%
 tab_options(table.font.size = 40) %>%
 fmt_number(decimals = 2)
```

term	estimate	std.error	statistic	p.value
(Intercept)	50.93	2.66	19.14	0.00
female_literacy_rate_2011	0.23	0.03	7.38	0.00



$$\widehat{Y} = \widehat{\beta}_0 + \widehat{\beta}_1 \cdot X$$

$$\widehat{\text{life expectancy}} = 50.9 + 0.232 \cdot \text{female literacy rate}$$

Fitted line is derived from the population SLR model

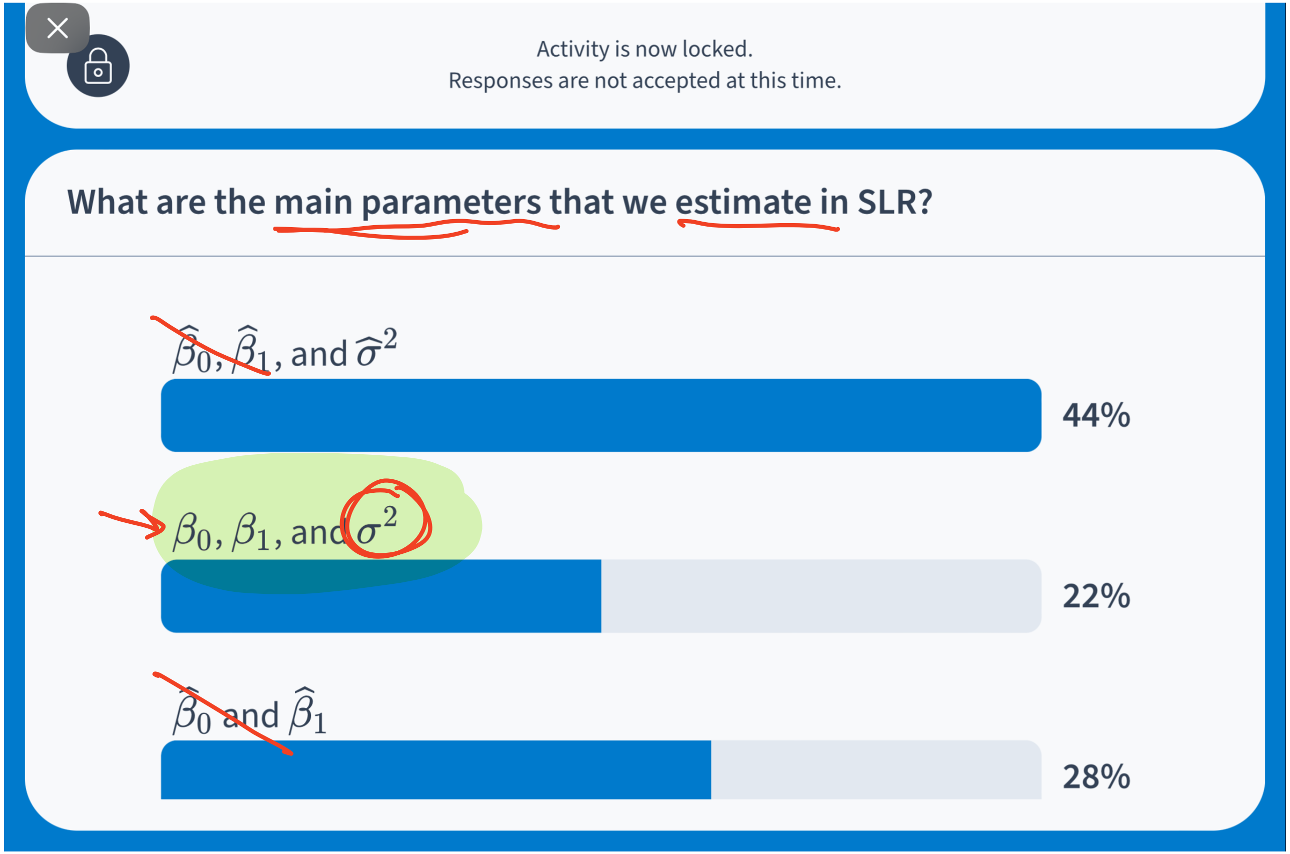
The (population) regression model is denoted by:

$$Y = \beta_0 + \beta_1 X + \epsilon$$

- β_0 and β_1 are **unknown** population parameters
- ϵ (epsilon) is the error about the line
  - It is assumed to be a random variable with a...
    - Normal distribution with mean 0 and constant variance σ^2
    - i.e. $\epsilon \sim N(0, \sigma^2)$

$\hat{\sigma}^2$

Poll Everywhere Question 1



$\beta_0 \quad \beta_1$

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1. Estimate the variance of the residuals

2. Using a hypothesis test, determine if there is enough evidence that population slope β_1 is not 0 (applies to β_0 as well)

3. Calculate and report the estimate and confidence interval for the population slope β_1 (applies to β_0 as well)

4. Calculate and report the estimate and confidence interval for the expected/mean response given X

$\widehat{\sigma}^2$: Needed ingredient for inference

- Recall our population model residuals are distributed by $\epsilon \sim N(0, \sigma^2)$
  - And our estimated residuals are $\hat{\epsilon} \sim N(0, \widehat{\sigma}^2)$
- Hence, the *variance* of the errors (residuals) is estimated by $\widehat{\sigma}^2$

$$\widehat{\sigma}^2 = S^2_{y|x} = \boxed{\frac{1}{n-2}} \sum_{i=1}^{n} \underbrace{(Y_i - \widehat{Y}_i)}_{\varepsilon_i}{}^2 = \boxed{\frac{1}{n-2}} SSE = MSE$$

$\widehat{\sigma}^2$: I hope R can calculate that for me...

- The *standard deviation* $\widehat{\sigma}$ is given in the R output as the `Residual standard error`
  - 4^{th} line from the bottom in the `summary()` output of the model:

glance(model1)

```
1  summary(model1)
```

$sigma

```
Call:
lm(formula = life_expectancy_years_2011 ~ female_literacy_rate_2011,
    data = gapm)

Residuals:
    Min      1Q  Median      3Q     Max 
-22.299  -2.670   1.145   4.114   9.498 

Coefficients:
```

6.142

```
1  # number of observations (pairs of data) used to run the model
2  nobs(model1)
```

```
[1] 80
```

$\widehat{\sigma}^2$ to SSE

- Recall how we minimized the SSE to find our line of best fit
- SSE and $\widehat{\sigma}^2$ are closely related:

$$\widehat{\sigma}^2 = \frac{1}{n-2} SSE$$

$$6.142^2 = \frac{1}{80-2} SSE$$

$$SSE = 78 \cdot 6.142^2 = 2942.48$$

$\hat{\varepsilon} \sim N(0, 6.142)$ $\hat{\sigma} =$

$N(\mu, \sigma^2)$

- $\boxed{\text{2942.48 is}}$ the smallest sums of squares of all possible regression lines through the data

Learning Objectives

1. Estimate the variance of the residuals
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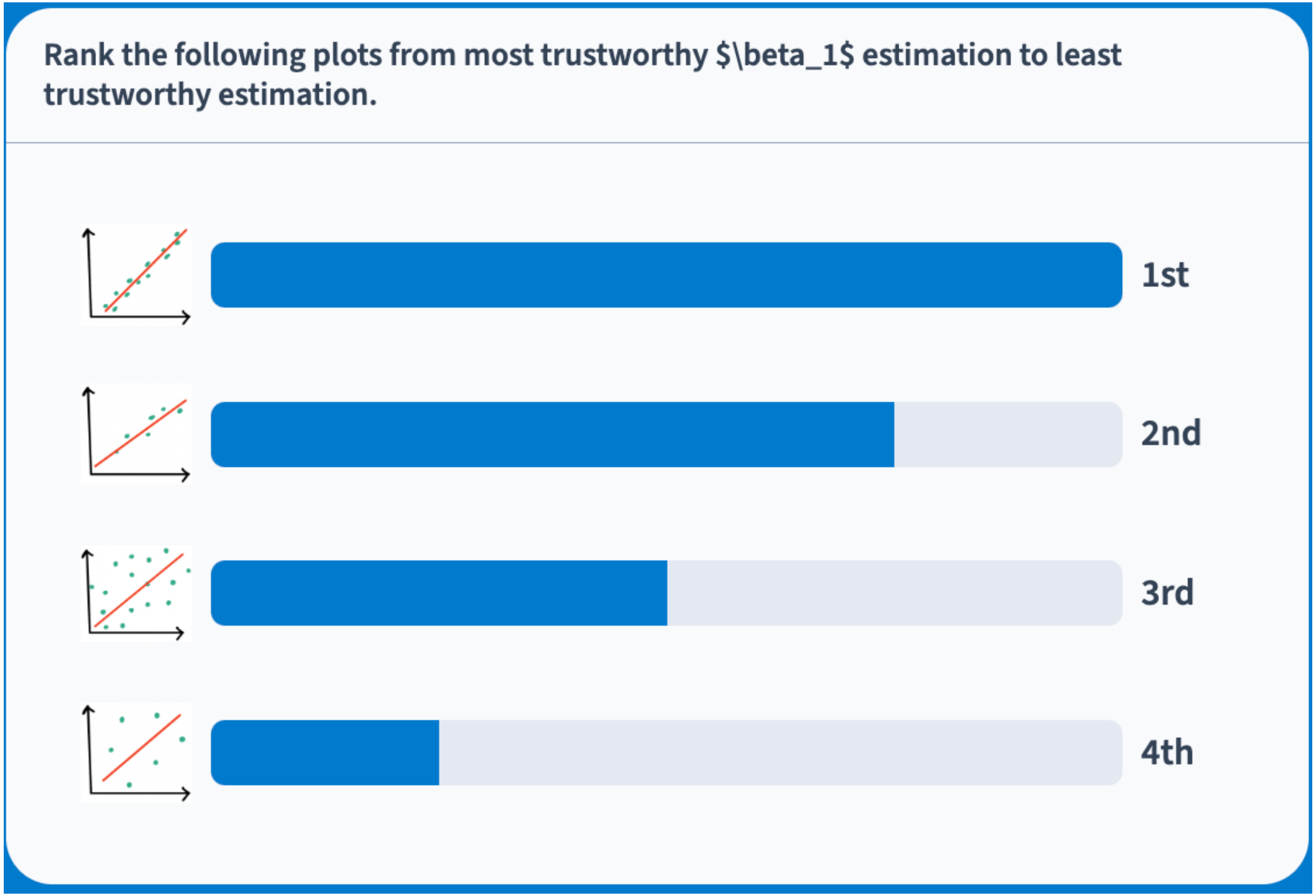
4. Calculate and report the estimate and confidence interval for the expected/mean response given X

Do we trust our estimate $\widehat{\beta}_1$?

estimate of β_1

- So far, we have shown that we think the estimate is 0.232
- $\widehat{\beta}_1$ uses our sample data to estimate the population parameter β_1
- Inference helps us figure out mathematically how much we trust our ~~best-fit line~~ the slope of the line
- Are we certain that the relationship between X and Y that we estimated reflects the true, underlying relationship?

Poll Everywhere Question 2



Inference for the population slope: hypothesis test and CI

Population model

line + random "noise"

$$Y = \beta_0 + \beta_1 \cdot X + \varepsilon$$

with $\varepsilon \sim N(0, \sigma^2)$
σ^2 is the variance of the residuals

Sample best-fit (least-squares) line

$$\widehat{Y} = \widehat{\beta}_0 + \widehat{\beta}_1 \cdot X$$

Note: Some sources use b instead of $\widehat{\beta}$

We have two options for inference:

1. Conduct the **hypothesis test**

$$H_0 : \beta_1 = 0$$
$$\text{vs. } H_A : \beta_1 \neq 0$$

Note: R reports p-values for 2-sided tests

2. Construct a **95% confidence interval** for the **population slope** β_1

Learning Objectives

1. Estimate the variance of the residuals

2. Using a hypothesis test, determine if there is enough evidence that population slope β_1 is not 0 (applies to β_0 as well)

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Steps in hypothesis testing

1. Check the assumptions regarding the properties of the underlying variable(s) being measured that are needed to justify use of the testing procedure under consideration.
2. State the null hypothesis H_0 and the alternative hypothesis H_A.
3. Specify the significance level α.
4. Specify the test statistic to be used and its distribution under H_0.

Critical region method

5. Form the decision rule for rejecting or not rejecting H_0 (i.e., specify the rejection and nonrejection regions for the test, based on both H_A and α).
6. Compute the value of the test statistic from the observed data.

p-value method

5. Compute the value of the test statistic from the observed data.
6. Calculate the p-value

7. Draw conclusions regarding rejection or nonrejection of H0.

General steps for hypothesis test for population slope β_1

$t \sim$ t-dist'n

1. For today's class, we are assuming that we have met the underlying assumptions (checked in our Model Evaluation step)

2. State the null hypothesis.

Often, we are curious if the coefficient is 0 or not:

$$H_0: \beta_1 = 0$$
$$\text{vs. } H_A: \beta_1 \neq 0$$

3. Specify the significance level.

Often we use $\alpha = 0.05$

4. Specify the test statistic and its distribution under the null

The test statistic is t, and follows a Student's t-distribution.

5. Compute the value of the test statistic

The calculated **test statistic** for $\widehat{\beta}_1$ is

under null

$$\boxed{t^* = \frac{\widehat{\beta}_1 - \beta_1}{\mathrm{SE}_{\widehat{\beta}_1}} = \frac{\widehat{\beta}_1}{\mathrm{SE}_{\widehat{\beta}_1}}}$$

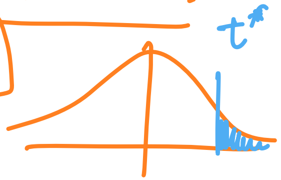
when we assume $H_0: \beta_1 = 0$ is true.

$$2 \cdot P(t > t^*)$$
$$= P(t > t^*) + P(t < -t^*)$$

6. Calculate the p-value

$2\times$ $P(t > t^*)$

We are generally calculating: $P(t > t*)$



7. Write conclusion for hypothesis test

We (reject/fail to reject) the null hypothesis that the slope is 0 at the $100\alpha\%$ significiance level. There is (sufficient/insufficient) evidence that there is significant association between (Y) and (X) (p-value $= 2P(t > t*)$).

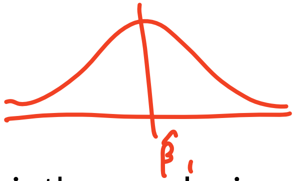
p-val ≥ 0.05

if p-value < 0.05

Standard error of fitted slope $\widehat{\beta}_1$

$$\text{SE}_{\widehat{\beta}_1} = \frac{s_{\text{residuals}}}{s_x\sqrt{n-1}}$$

$\text{SE}_{\widehat{\beta}_1}$ is the **variability** of the statistic $\widehat{\beta}_1$



variance

- $s^2_{\text{residuals}}$ is the sd of the residuals
- s_x is the sample sd of the explanatory variable x
- n is the sample size, or the number of (complete) pairs of points

σ is sd

σ^2 var

se of coef estimate is inversely prop to x sd

Calculating standard error for $\widehat{\beta}_1$ (1/2)

- **Option 1:** Calculate using the formula

```
glance(model1)
```

```
# A tibble: 1 × 12
  r.squared adj.r.squared sigma statistic  p.value    df logLik   AIC   BIC
      <dbl>         <dbl> <dbl>     <dbl>    <dbl> <dbl>  <dbl> <dbl> <dbl>
1     0.411         0.403  6.14      54.4 1.50e-10     1  -258.  521.  529.
# ℹ 3 more variables: deviance <dbl>, df.residual <int>, nobs <int>
```

```
# standard deviation of the residuals (Residual standard error in summary() output)
(s_resid <- glance(model1)$sigma)
```

```
[1] 6.142157
```

$\rightarrow sd_{residual} = \hat{\sigma}$

```
# standard deviation of x's
(s_x <- sd(gapm$female_literacy_rate_2011, na.rm=T))
```

```
[1] 21.95371
```

$= S_x$

```
# number of pairs of complete observations
(n <- nobs(model1))
```

```
[1] 80
```

n

```
(se_b1 <- s_resid/(s_x * sqrt(n-1))) # compare to SE in regression output
```

```
[1] 0.03147744
```

$\hookrightarrow \dfrac{S_{resid}}{S_x\sqrt{n-1}}$

Calculating standard error for $\widehat{\beta}_1$ (2/2)

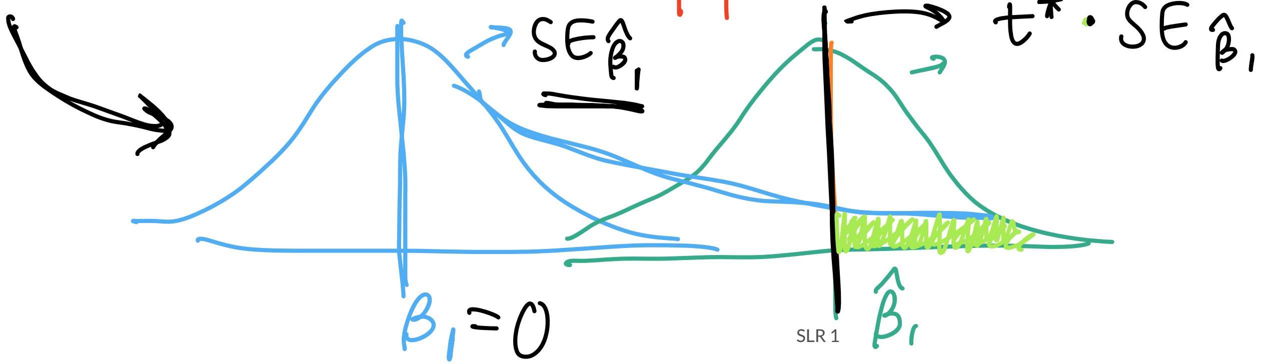
- **Option 2:** Use regression table

```r
# recall model1_b1 is regression table restricted to b1 row
model1_b1 <-tidy(model1) %>% filter(term == "female_literacy_rate_2011")
model1_b1 %>% gt() %>%
  tab_options(table.font.size = 45) %>% fmt_number(decimals = 4)
```

term	estimate	std.error	statistic	p.value
female_literacy_rate_2011	0.2322	$\boxed{0.0315}$	7.3766	0.0000

$$SE_{\hat{\beta}_1} = 0.0315$$

$$\boxed{t^* = \frac{\hat{\beta}_1}{SE_{\hat{\beta}_1}}}$$



Some important notes

- Today we are discussing the hypothesis test for a **single** coefficient

- The test statistic for a single coefficient follows a Student's t-distribution

  - It can also follow an F-distribution, but we will discuss this more with multiple linear regression and multi-level categorical covariates

- Single coefficient testing can be done on any coefficient, but it is most useful for continuous covariates or binary covariates

  - This is because testing the single coefficient will still tell us something about the overall relationship between the covariate and the outcome

  - We will talk more about this with multiple linear regression and multi-level categorical covariates

Poll Everywhere Question 3

t-test:

$\beta_0 = 0$

vs. $\beta_0 \neq 0$

breaks:

$\rightarrow$ $\underline{\beta_0 = \beta_1 = 0}$

$\underline{\text{vs.}}$ $\beta_0 \neq 0$ or

$\beta_1 \neq 0$

Life expectancy example: hypothesis test for population slope β_1 (1/4)

- Steps 1-4 are setting up our hypothesis test: not much change from the general steps

1. For today's class, we are assuming that we have met the underlying assumptions (checked in our Model Evaluation step)

2. State the null hypothesis.

We are testing if the slope is 0 or not:

$$\begin{aligned} H_0 &: \beta_1 = 0 \\ \text{vs. } H_A &: \beta_1 \neq 0 \end{aligned}$$

3. Specify the significance level.

Often we use $\alpha = 0.05$

4. Specify the test statistic and its distribution under the null

The test statistic is t, and follows a Student's t-distribution.

Life expectancy example: hypothesis test for population slope β_1 (2/4)

5. Compute the value of the test statistic

$t^* = \dfrac{\hat\beta_1 - \beta_1}{SE_{\hat\beta_1}}$

- **Option 1:** Calculate the test statistic using the values in the regression table

```
# recall model1_b1 is regression table restricted to b1 row
model1_b1 <-tidy(model1) %>% filter(term == "female_literacy_rate_2011")
model1_b1 %>% gt() %>%
  tab_options(table.font.size = 40) %>% fmt_number(decimals = 2)
```

term	estimate	std.error	statistic	p.value
female_literacy_rate_2011	0.23	0.03	7.38	0.00

```
(TestStat_b1 <- model1_b1$estimate / model1_b1$std.error)
```

```
[1] 7.376557
```

- **Option 2:** Get the test statistic value (t^*) from R

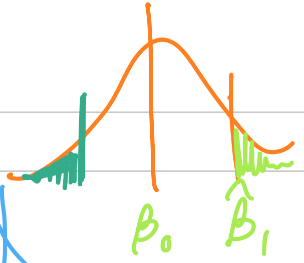
```
model1_b1 %>% gt() %>%
  tab_options(table.font.size = 40) %>% fmt_number(decimals = 2)
```

term	estimate	std.error	statistic	p.value
female_literacy_rate_2011	0.23	0.03	7.38	0.00

Life expectancy example: hypothesis test for population slope β_1 (3/4)

6. Calculate the p-value

- The p-value is the *probability of obtaining a test statistic* **just as extreme or more extreme** *than the* ***observed*** *test statistic assuming the null hypothesis H_0 is true* → $\beta_1 = 0$ t^*
- We know the probability distribution of the test statistic (the *null distribution*) assuming H_0 is true
- Statistical theory tells us that the test statistic t can be modeled by a t-distribution with $df = n - 2$.
  - We had 80 countries' data, so $n = 80$
- **Option 1:** Use `pt()` and our calculated test statistic




```
(pv = 2*pt(TestStat_b1, df=80-2, lower.tail=F))
```

```
[1] 1.501286e-10
```

T

- **Option 2:** Use the regression table output

```
model1_b1 %>% gt() %>%
  tab_options(table.font.size = 40)
```

term	estimate	std.error	statistic	p.value
female_literacy_rate_2011	0.2321951	0.03147744	7.376557	1.501286e-10

Life expectancy example: hypothesis test for population slope β_1 (4/4)

7. Write conclusion for the hypothesis test

We reject the null hypothesis that the slope is 0 at the 5% significance level. There is sufficient evidence that there is significant association between female life expectancy and female literacy rates (p-value < 0.0001).

Note on hypothesis testing using R

- We can basically skip Step 5 if we are using the "Option 2" route

- In our assignments: if you use Option 2, Step 5 is optional
  - Unless I specifically ask for the test statistic!!

REFERENCE

Life expectancy ex: hypothesis test for population intercept β_0 (1/4)

- Steps 1-4 are setting up our hypothesis test: not much change from the general steps

1. For today's class, we are assuming that we have met the underlying assumptions (checked in our Model Evaluation step)

2. State the null hypothesis.

We are testing if the intercept is 0 or not:

$$\begin{aligned} H_0 &: \beta_0 = 0 \\ \text{vs. } H_A &: \beta_0 \neq 0 \end{aligned}$$

3. Specify the significance level

Often we use $\alpha = 0.05$

4. Specify the test statistic and its distribution under the null

This is the same as the slope. The test statistic is t, and follows a Student's t-distribution.

REFERENCE

Life expectancy ex: hypothesis test for population intercept β_0 (2/4)

5. Compute the value of the test statistic

- **Option 1:** Calculate the test statistic using the values in the regression table

```r
# recall model1_b1 is regression table restricted to b1 row
model1_b0 <-tidy(model1) %>% filter(term == "(Intercept)")
model1_b0 %>% gt() %>%
  tab_options(table.font.size = 40) %>% fmt_number(decimals = 2)
```

term	estimate	std.error	statistic	p.value
(Intercept)	50.93	2.66	19.14	0.00

```r
(TestStat_b0 <- model1_b0$estimate / model1_b0$std.error)
```

```
[1] 19.1429
```

- **Option 2:** Get the test statistic value (t^*) from R

```r
model1_b0 %>% gt() %>%
  tab_options(table.font.size = 40) %>% fmt_number(decimals = 2)
```

term	estimate	std.error	statistic	p.value
(Intercept)	50.93	2.66	19.14	0.00

REFERENCE

Life expectancy ex: hypothesis test for population intercept β_0 (3/4)

6. Calculate the p-value

- **Option 1:** Use `pt()` and our calculated test statistic

```
(pv = 2*pt(TestStat_b0, df=80-2, lower.tail=F))
```

```
[1] 3.325312e-31
```

- **Option 2:** Use the regression table output

```
model1_b0 %>% gt() %>%
  tab_options(table.font.size = 40)
```

term	estimate	std.error	statistic	p.value
(Intercept)	50.9279	2.660407	19.1429	3.325312e-31

Life expectancy ex: hypothesis test for population intercept β_0 (4/4)

7. Write conclusion for the hypothesis test

We reject the null hypothesis that the intercept is 0 at the 5% significance level. There is sufficient evidence that the intercept for the association between average female life expectancy and female literacy rates is different from 0 (p-value < 0.0001).

- Note: if we fail to reject H_0, then we could decide to remove the intercept from the model to force the regression line to go through the origin (0,0) if it makes sense to do so for the application.

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1. Estimate the variance of the residuals
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4. Calculate and report the estimate and confidence interval for the expected/mean response given X

Inference for the population slope: hypothesis test and CI

Population model

line + random "noise"

$$Y = \beta_0 + \beta_1 \cdot X + \varepsilon$$

with $\varepsilon \sim N(0, \sigma^2)$
σ^2 is the variance of the residuals

Sample best-fit (least-squares) line

$$\widehat{Y} = \widehat{\beta}_0 + \widehat{\beta}_1 \cdot X$$

Note: Some sources use b instead of $\widehat{\beta}$

We have two options for inference:

1. Conduct the **hypothesis test**

$$\begin{aligned} H_0 &: \beta_1 = 0 \\ \text{vs. } H_A &: \beta_1 \neq 0 \end{aligned}$$

Note: R reports p-values for 2-sided tests

2. Construct a **95% confidence interval** for the **population slope** β_1

Confidence interval for population slope β_1

Recall the general CI formula:

critical value

$$\widehat{\beta}_1 \pm t_{\alpha,n-2} \cdot SE_{\widehat{\beta}_1}$$

To construct the confidence interval, we need to:

- Set our α-level
- Find $\widehat{\beta}_1$
- Calculate the t^*_{n-2}
- Calculate $SE_{\widehat{\beta}_1}$

Calculate CI for population slope β_1 (1/2)

$$\widehat{\beta}_1 \pm t^* \cdot SE_{\beta_1}$$

where t^* is the t-distribution critical value with $df = n - 2$.

- **Option 1:** Calculate using each value

Save values needed for CI:

```
b1 <- model1_b1$estimate
SE_b1 <- model1_b1$std.error
```

```
nobs(model1) # sample size n
```

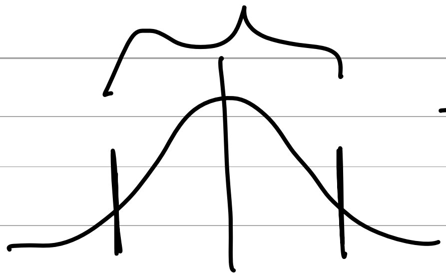
```
[1] 80
```

dash

```
(tstar <- qt(.975, df = 80-2))
```

```
[1] 1.990847
```

95% CI



$t_{\alpha=0.05,\, n-2=78}$

Use formula to calculate each bound

```
(CI_LB <- b1 - tstar*SE_b1)
```

```
[1] 0.1695284
```

```
(CI_UB <- b1 + tstar*SE_b1)
```

```
[1] 0.2948619
```

Calculate CI for population slope β_1 (2/2)

$$\widehat{\beta}_1 \pm t^* \cdot SE_{\beta_1}$$

where t^* is the t-distribution critical value with $df = n - 2$.

- **Option 2:** Use the regression table

```
tidy(model1, conf.int = T) %>% gt() %>%
  tab_options(table.font.size = 40) %>% fmt_number(decimals = 3)
```

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	50.928	2.660	19.143	0.000	45.631	56.224
female_literacy_rate_2011	0.232	0.031	7.377	0.000	0.170	0.295

$t \sim t\text{-dist}$

test statistic → $\boxed{\text{t-value}}$ → in dist.

t-value → $\boxed{\text{t-value}}$

critical value

→ corresponds to desired α - level

Reporting the coefficient estimate of the population slope

- When we report our results to someone else, we don't usually show them our full hypothesis test
  - In an informal setting, someone may want to see it
- Typically, we report the estimate with the confidence interval
  - From the confidence interval, your audience can also deduce the results of a hypothesis test
- Once we found our CI, we often just write the interpretation of the coefficient estimate:

General statement for population slope inference

For every increase of 1 unit in the X-variable, there is an expected average increase of $\widehat{\beta}_1$ units in the Y-variable (95%: LB, UB).

- **In our example:** For every 1% increase in female literacy rate, the average life expectancy is expected to increase, on average, 0.232 years (95% CI: 0.170, 0.295).

95% CI for population param slope

Poll Everywhere Question 4

Based off the following regression table, write a statement for the intercept's interpretation. Make sure to include the confidence interval.

Poll_Ev_Q2.png

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	50.928	2.660	19.143	0.000	45.631	56.224
female_literacy_rate_2011	0.232	0.031	7.377	0.000	0.170	0.295

Nobody has responded yet.

Hang tight! Responses are coming in.

For reference: quick CI for β_0

critical value

CI: $(\hat{\beta}_0 - t^* SE_{\hat{\beta}_0}, \hat{\beta}_0 + t^* SE_{\hat{\beta}_0})$

LB UB

- Calculate CI for population **intercept** β_0: $\widehat{\beta}_0 \pm t^* \cdot SE_{\hat{\beta}_0}$

where t^* is the t-distribution critical value with $df = n - 2$

- Use the regression table

?tidy alpha= 0.05

```
tidy(model1, conf.int = T) %>% gt() %>%
  tab_options(table.font.size = 40) %>% fmt_number(decimals = 3)
```

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	50.928	2.660	19.143	0.000	45.631	56.224
female_literacy_rate_2011	0.232	0.031	7.377	0.000	0.170	0.295

General statement for population intercept inference

The expected outcome for the Y-variable is ($\widehat{\beta}_0$) when the X-variable is 0 (95% CI: LB, UB).

- **For example:** The expected/average life expectancy is 50.9 years when the female literacy rate is 0 (95% CI: 45.63, 56.22).

Learning Objectives

1. Estimate the variance of the residuals
2. Using a hypothesis test, determine if there is enough evidence that population slope β_1 is not 0 (applies to β_0 as well)
3. Calculate and report the estimate and confidence interval for the population slope β_1 (applies to β_0 as well)
4. Calculate and report the estimate and confidence interval for the expected/mean response given X

Finding a mean response given a value of our independent variable

term	estimate	std.error	statistic	p.value
(Intercept)	50.928	2.660	19.143	0.000
female_literacy_rate_2011	0.232	0.031	7.377	0.000

$$\underbrace{\widehat{\text{life expectancy}}}_{\hat{Y}} = 50.9 + 0.232 \cdot \underbrace{\text{female literacy rate}}_{X}$$

$\hat{Y} \mid X$ and its CI

- What is the expected/predicted life expectancy for a country with female literacy rate 60%?

average

$$\widehat{\text{life expectancy}} = 50.9 + 0.232 \cdot 60 = 64.82$$

```
(y_60 <- 50.9 + 0.232*60)
```

```
[1] 64.82
```

- How do we interpret the expected value?
  - We sometimes call this “predicted” value, since we can technically use a literacy rate that is not in our sample
- How variable is it?

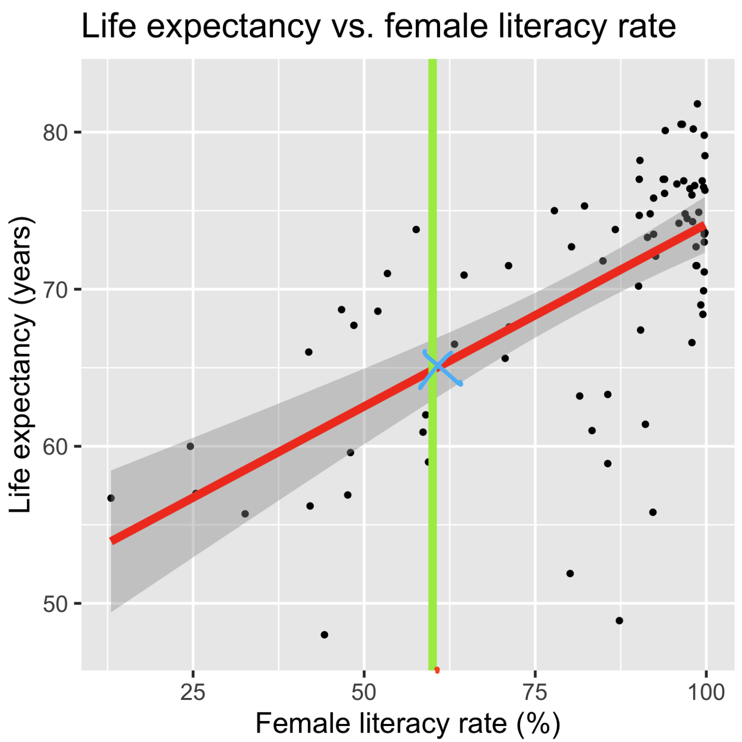
Mean response/prediction with regression line

Recall the population model:

line + random "noise"

$$Y = \beta_0 + \beta_1 \cdot X + \varepsilon$$

with $\varepsilon \sim N(0, \sigma^2)$



- When we take the expected value, at a given value X^*, the average expected response at X^* is:

$$\widehat{E}[Y|X^*] = \widehat{\beta}_0 + \widehat{\beta}_1 X^*$$

- These are the points on the regression line
- The mean responses have variability, and we can calculate a CI for it, for every value of X^*

CI for population mean response ($E[Y|X^*]$ or $\mu_{Y|X^*}$)

$\hat{E}[Y|X]$

$\hat{Y}|X$

$E(Y|X)$

95%

$$\widehat{E}[Y|X^*] \pm t^*_{n-2} \cdot \boxed{SE_{\widehat{E}[Y|X^*]}}$$

$$SE_{\widehat{E}[Y|X^*]} = s_{\text{residuals}} \sqrt{\frac{1}{n} + \frac{(X^* - \overline{X})^2}{(n-1)s_X^2}}$$

- $\widehat{E}[Y|X^*]$ is the predicted value at the specified point X^* of the explanatory variable
- $s^2_{\text{residuals}}$ is the sd of the residuals
- n is the sample size, or the number of (complete) pairs of points
- $\overline{X}$ is the sample mean of the explanatory variable x
- s_X is the sample sd of the explanatory variable X

- Recall that t^*_{n-2} is calculated using `qt()` and depends on the confidence level $(1-\alpha)$

Example Option 1: CI for mean response $\mu_{Y|X^*}$

$\hat{Y}|X$ $\quad var(\hat{Y}|X)$

$= var(\hat\beta_0 + \hat\beta_1 X)$

$= var(\hat\beta_0) + var(\hat\beta_1 X)$

Find the 95% CI for the mean life expectancy when the female literacy rate is 60.

$$\widehat{E}[Y|X^*] \pm t^*_{n-2} \cdot SE_{\widehat{E}[Y|X^*]}$$

$$64.8596 \pm 1.990847 \cdot s_{residuals}\sqrt{\frac{1}{n} + \frac{(X^* - \bar{x})^2}{(n-1)s_x^2}}$$

$$64.8596 \pm 1.990847 \cdot 6.142157\sqrt{\frac{1}{80} + \frac{(60 - 81.65375)^2}{(80-1)21.95371^2}}$$

$$64.8596 \pm 1.990847 \cdot 0.9675541$$

$$64.8596 \pm 1.926252$$

$$(62.93335, 66.78586)$$

```
(Y60 <- 50.9278981 + 0.2321951 * 60)
```

```
[1] 64.8596
```

```
(tstar <- qt(.975, df = 78))
```

```
[1] 1.990847
```

```
(s_resid <- glance(model1)$sigma)
```

```
[1] 6.142157
```

```
(n <- nobs(model1))
```

```
[1] 80
```

```
(mx <- mean(gapm$female_literacy_rate_2011,
```

```
[1] 81.65375
```

na.rm=T)

```
(s_x <- sd(gapm$female_literacy_rate_2011, r
```

```
[1] 21.95371
```

```
(SE_Yx <- s_resid *sqrt(1/n + (60 - mx)^2/((n-1)*s_x^2)))
```

```
[1] 0.9675541
```

```
(MOE_Yx <- SE_Yx*tstar)
```

```
[1] 1.926252
```

LB

```
Y60 - MOE_Yx
```

```
[1] 62.93335
```

UB

```
Y60 + MOE_Yx
```

```
[1] 66.78586
```

Example Option 2: CI for mean response $\mu_{Y|X^*}$

Find the 95% CI's for the mean life expectancy when the female literacy rate is 60 and 80.

- Use the base R `predict()` function
- Requires specification of a `newdata` "value"
  - The `newdata` value is X^*
  - This has to be in the format of a data frame though
  - with column name identical to the predictor variable in the model

```
newdata <- data.frame(female_literacy_rate_2011 = c(60, 80))
newdata
```

X
variable name
data

```
  female_literacy_rate_2011
1                        60
2                        80
```

```
predict(model1,
        newdata=newdata,
        interval="confidence")
```

model1 ← lm(...)

?predict

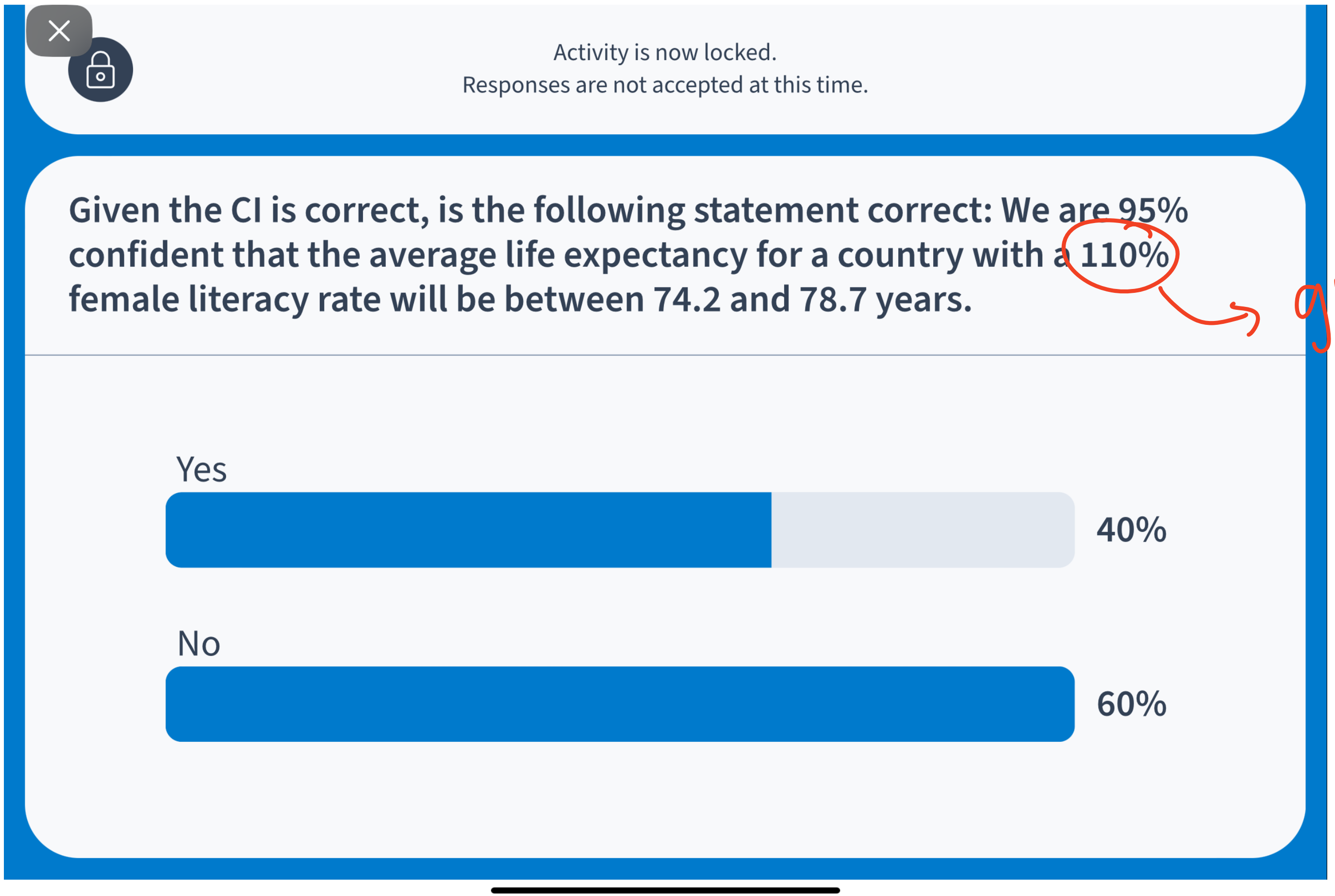
```
       fit      lwr      upr
1 64.85961 62.93335 66.78586
2 69.50351 68.13244 70.87457
```

60
80

Interpretation 60%

We are 95% confident that the **average** life expectancy for a country with a 60% female literacy rate will be between 62.9 and 66.8 years.

Poll Everywhere Question 5

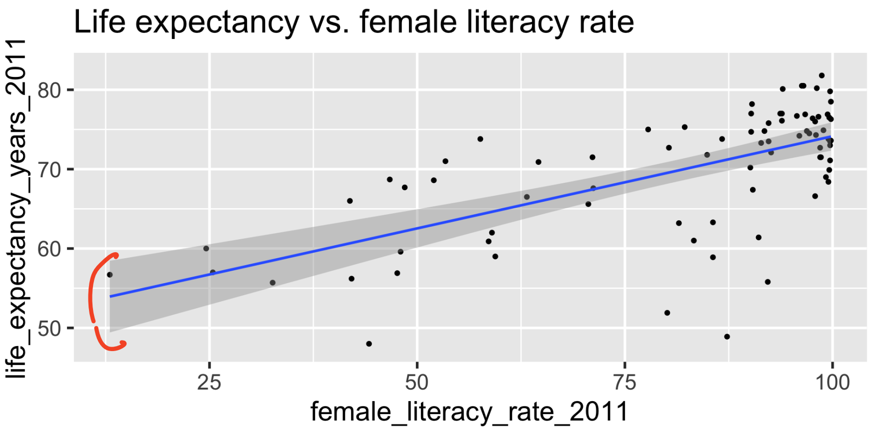


greater than 100%
not possible

Confidence bands for mean response $\mu_{Y|X^*}$

- Often we plot the CI for many values of X, creating **confidence bands**
- The confidence bands are what ggplot creates when we set `se = TRUE` within `geom_smooth`
- Think about it: for what values of X are the confidence bands (intervals) narrowest?

```
ggplot(gapm,
       aes(x=female_literacy_rate_2011,
           y=life_expectancy_years_2011)) +
  geom_point()+
  geom_smooth(method = lm, se=TRUE)+
  ggtitle("Life expectancy vs. female literacy rate")
```



Width of confidence bands for mean response $\mu_{Y|X^*}$

- For what values of X^* are the confidence bands (intervals) narrowest? widest?

$$\widehat{E}[Y|X^*] \pm t^*_{n-2} \cdot SE_{\widehat{E}[Y|X^*]}$$

$$\widehat{E}[Y|X^*] \pm t^*_{n-2} \cdot s_{residuals} \sqrt{\frac{1}{n} + \frac{(X^* - \bar{x})^2}{(n-1)s_x^2}}$$

narrowest @ purple line (mean x)